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Methods for studying ferroresonance in pi and series power circuits

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METHODS FOR STUDYING FERRORESONANCE IN PI AND SERIES POWER CIRCUITS

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Methods for studying ferroresonance in

pi and series power circuits

by

B. Peter Daay

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TABLE OF CONTENTS

LIST OF SYMBOLS AND DEFINITIONS

m_1	Slope of the unsaturated region of the magnetization curve
m_2	Slope of the saturated region of the magnetization curve
λ_0	Flux at the saturation knee of the curve
I_0	Current at the saturation knee of the curve
λ_m	Flux input amplitude to the nonlinearity
I_m	Current input amplitude to the nonlinearity
$\lambda(t)$	Instantaneous input flux to the nonlinearity
L_{eq}	Equivalent linear inductance
$N(\lambda_m)$	Describing Function of the power transformer in a pi circuit
$\dot{N}(\lambda_m)$	Describing Function derivative with respect to λ_m
$N_p(\lambda_m)$	In-phase component of the Describing Function in a pi circuit
$N_q(\lambda_m)$	Quadrature component of the Describing Function in a pi circuit
$N(\lambda_m,\phi)$	Incremental Input Describing Function for a pi circuit
$N(I_m,\phi)$	Incremental Input Describing Function for a series circuit
$N(I_m)$	Describing Function of the power transformer in a series circuit
$\dot{N}(I_m)$	Describing Function derivative with respect to I_m
$N_p(I_m)$	In-phase component of the Describing Function in a series circuit

$N_q(I_m)$	Quadrature component of the Describing Function in a series circuit
μ	Parameter
$i(t)$	Total instantaneous current
$i_c(t)$	Instantaneous capacitance current
$i_T(t)$	Instantaneous magnetization current
$I_T(s)$	Transform of the fundamental frequency of the current
$E(s)$	Transform of the sinusoidal input voltage
$G_p(s)$	Linear part representation of the nonlinear system in pi circuit
$G_s(s)$	Linear part representation of the nonlinear system in series circuit
K_1	Real part of the 60 Hertz point in a pi circuit
K_2	Imaginary part of the 60 Hertz point in a pi circuit
K_3	Real part of the 60 Hertz point in a series circuit
K_4	Imaginary part of the 60 Hertz point in a series circuit
LM	Linear mode
NLM	Nonlinear mode

I. INTRODUCTION

A. Basis for the Work

Fundamental ferroresonance is a jump resonance characterized by a random over-voltage that may occur in situations where a modest size capacitor is connected either in parallel or series with an iron cored transformer.

This undesirable random over-voltage across the transformer (Figures 1 and 2) causes overheating and insulation damage to the transformer or customer appliances and may result in injuries to utility personnel.

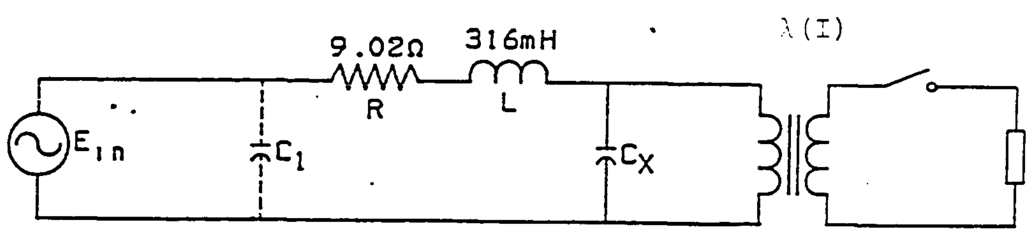


Figure 1. Pi circuit configuration

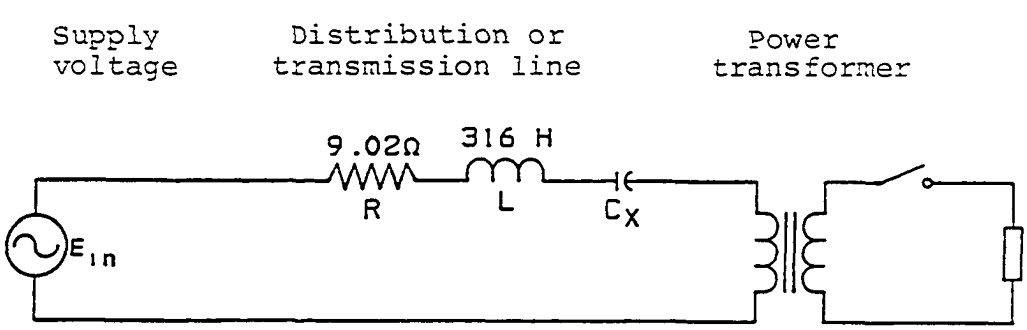


Figure 2. Series circuit configuration

There are several trends in power system design that may contribute to the increasing frequency of ferroresonance phenomenon (1). Specifically, these trends in distribution and transmission systems are:

1. Increased use of underground cable in primary circuits of the distribution system resulting in much higher circuit shunt capacitance than in overhead lines.

2. Increasing distribution and transmission voltage levels.

3. Improved transformer material and design resulting in transformers with lower values of exciting current, higher values of magnetizing reactance, and higher values of shunt capacitance.

4. Shunt compensation in transmission lines.

There is increasing evidence that the choice of the magnetization characteristic model for any of the circuits could influence the accuracy of the results of ferroresonance studies (2). Therefore, mathematical modeling of the magnetization curve of a power transformer appears to be important in the evaluation of the possible occurrence of ferroresonance (2).

There are a number of methods available in the current literature that deal with the problem of ferroresonance for unloaded circuit conditions. However, they can basically be categorized into two groups: (1) the Describing Function techniques, and (2) Piecewise Linearization methods.

In addition to these procedures, a digital computer program package developed by C. W. Gear can also be applied to the analysis of ferroresonance (3).

It is evident that previous solutions provided by these methods were all limited and limiting studies of ferroresonance of the power system.

B. Objectives and Uniqueness of this Study

This discussion is concerned with the development of a practical tool to predict ferroresonance in pi and series power circuits that are left complex to be more representative of real situations and without the restriction to a specific saturation curve of power transformers. The study examines thoroughly all the parameters affecting the occurrence of ferroresonance to establish the limits of circuit susceptibility and to determine the necessary analytical conditions to turn the phenomenon into a stable mode. It is believed that for the first time (1) the Incremental Describing Function has been applied to study ferroresonance in the series circuit configuration, and (2) the Gear's digital computer program is used to solve the ferroresonance problem.

Also, the methods presented differ from previous work in that they introduce the application of the Describing Function based on two slopes of the magnetization curve to find analytical solutions for cases where the graphical technique of G. W. Swift and other methods are either inadequate or not applicable. This also facilitates the study of ferroresonance for loaded conditions of a general circuit. Simplification over existing methods will be even more pronounced for future studies when studying this phenomenon under different load composition and representation such as constant impedance, constant current, and constant MVA loads.

Even for the unloaded simplified circuit configurations, the labor involved in other methods is unnecessary. if the same information can be obtained more directly by a simple straightforward approach.

There also appears to be a lack of complete experimentation in the ferroresonance area. Although there has been some experimental work on simplified circuits, this study includes complete experiments on both pi and series circuit configurations. This effort should contribute to a deeper physical insight and better understanding of the phenomenon in power systems.

The objectives of this dissertation are:

1. To study fundamental ferroresonance in power circuits including one transformer with the following advantages:

a. Without the restriction to a specific transformer magnetization curve representation by nonlinear terms.
b. Provides a direct analytical solution without the need for a trial-and-error procedure.
c. Pertains to pi and series power circuits and applies to power and instrument potential transformers of the power system.
d. Applicable with equal facility to either loaded or unloaded circuits.

2. Review methods of solutions:
   a. Piecewise Linearization Technique
   b. G. W. Swift's method
   c. Gear's Digital program
   d. Study Methodology - proposed method of this study

   Compare the solution of these methods with the experimental results to determine the best approach to study ferroresonance in a given circuit configuration.

3. Some of the points of difference between previous methods and this study include determining a complete set of experimentally validated critical R, L, and C parameters to trigger the phenomenon and providing a basis for this study; conducting a set of experiments to verify the analytical prediction of ferroresonance and obtaining results that can be, in general, compared with the analytical solution of the other methods.

The work consists of the following parts:

1. Application of the Incremental Input Describing Function based on the two-slope approximation of the magnetization curve. The approximation of the magnetic characteristic of a power transformer is assumed to have three regions: (a) the first linear part is

in close agreement with the actual curve near the origin, (b) the second linear part, in close agreement with the actual curve for large current value, and (c) transitional region, that region between the other two which is modelled less accurately. The Describing Function using the above two linear slopes of the magnetic characteristic not only simplify the solution of the given practical circuits but also provides a better approximation for the transitional region than the two-slope piecewise linearization.

2. Review of other methods such as: (a) G. W. Swift's one-nonlinear term Incremental Describing Function, (b) piecewise linearization, and (c) Gear's ordinary differential equation problem solver. Comparisons of analytical solutions with experimental results will be accomplished in order to determine the best approach to the study of ferroresonance in typical circuits representing a power system.

3. Provide an alternative tool applicable to pi and series practical circuits to better understand the dynamics of the phenomenon.

4. Determination of a set of experimentally validated critical parameters to trigger the phenomenon.

5. Conducting a set of experiments to verify the analytical prediction of ferroresonance and to obtain results that can be compared with the analytical solutions obtained by various techniques for both pi and series circuits. Typical results of this study are:
   a. Range of R, L, and C producing ferroresonance
   b. Transformer critical voltage levels
   c. Critical input voltage levels
   d. Transient switching characteristics

A method has been presented in this study for the analysis of ferroresonance in the unloaded or loaded pi and series circuits configurations. In these circuits, the threshold of inductance, capacitance, and resistance for the onset of ferroresonance can be determined. Prediction can then be made about whether or not ferroresonance would persist once initiated and at what system voltage level it occurs spontaneously. This method provides some physical insight into the phenomenon because the influence of various parameters such as frequency, inductance, resistance, and capacitance on the multi-modal operation of these circuits can easily be investigated separately.

II. LITERATURE REVIEW

The main goal of ferroresonance studies is to determine whether or not ferroresonance will occur and the number of operational modes that exist in a series or parallel circuit. Because mathematical modeling of the magnetic characteristic of a power transformer plays a critical part in the outcome of the study, it is important in the evaluation of the occurrence of ferroresonance.

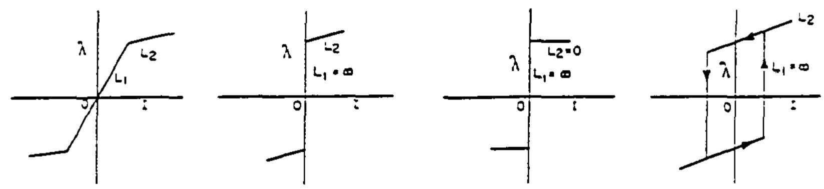
The interest in the phenomenon in electrical circuits can be traced back to the early years of the 20th century (4-5). In recent years, authors have recognized and reported the occurrence of ferroresonance behavior in EHV and distribution systems (6-10).

Many studies have reported on this subject and various approaches have been attempted to solve one or more aspects of the problem. Several researchers have used analytical techniques and developed various expressions to represent the nonlinear transformer magnetization characteristic and solved the resulting differential equation by classical methods (11-15). Others have used simulation methods and employed physical models (16,17). In addition, mathematical models have been developed for computations on both analogue and digital computers (18-20).

Several authors have used two straight lines to represent the magnetization curve of the iron-cored inductor. These studies were conducted between 1931 and 1978 and on a series circuit consisting of a resistor, capacitor, and an iron-cored inductor. While power transformers today are connected to either a transmission or a distribution line, implying the presence of an inductance in the transformer's circuit, these studies have excluded from their series circuit this inductance which influences the occurrence of ferroresonance. Also, their studies do not mention anything about the linear parameters of the systems R, L, and C, which are as important as the magnetization curve for the analysis of ferroresonance. In summary, the circuit for their studies did not represent a power circuit.

As early as 1931, Boyajian (21) derived the differential equations for the series circuit with the magnetization characteristic of the transformer shown in Figure 3a. However, the necessity to compute the solutions by trial and error with the means available at that time greatly limited the usefulness of his method.

In 1938, Travis and Weygandt (22), and in 1939, Travis (23) followed the same avenue as Boyajian, except that the transformer characteristic was represented as shown in Figure 3b, and they succeeded only in predicting the possi-



(a) (b) (c) (d)

Figure 3. Piecewise representations for magnetization curve

bility of subharmonic operation of the series circuit.

In 1959, Salihi (24), and in 1970, Kassakian and Wilson (25) adopted for their studies the transformer characteristic shown in Figure 3c and assumed that it could be replaced with a switch. Later, many authors emphasized that the characteristic of the modern power transformer is adequately represented by a quintic nonlinearity instead of a switch (26,27).

In recent years, emphasis has been placed on the use of the Describing Function technique which was developed from nonlinear control theory (2, 26-32) and employed in 1969 by Swift (26), in studying ferroresonance. His magnetic characteristic model consisted of a linear term plus a nonlinear quintic one.

Ashok Kumar et al. noted that Swift's graphical solution gives pessimistic results concerning the prediction of susceptibility of various power transformers to ferroresonance. Therefore, he intended to obtain analytically the lambda intersecting circles. Ashok plotted ferroresonant regions of different nonlinearities as shown in Figure 4 and noted that the envelope for the 7th degree nonlinearity is farthest from the real axis and that for a cubic is closest to it, therefore, the 7th degree nonlinearity representation of the fast saturation type of transformers in a given circuit is the most sensitive to the production of ferroresonance. Ashok also noted that his general nonlinearity envelope follows the cubic curve at (-1,0) point, crosses the 5th degree envelope and finally follows the 7th degree envelope.

In 1972, Ashok Kumar et al. (2) showed that the order of the nonlinearity determines the susceptibility of a system to ferroresonance. This implies that G. W. Swift's representation of the magnetization curve by one nonlinear quintic term limited the application to a specific type of transformer.

Kumar et al. also represented analytically the excitation characteristic by a cubic type nonlinearity and obtained the intersecting lambdas. However, he concluded

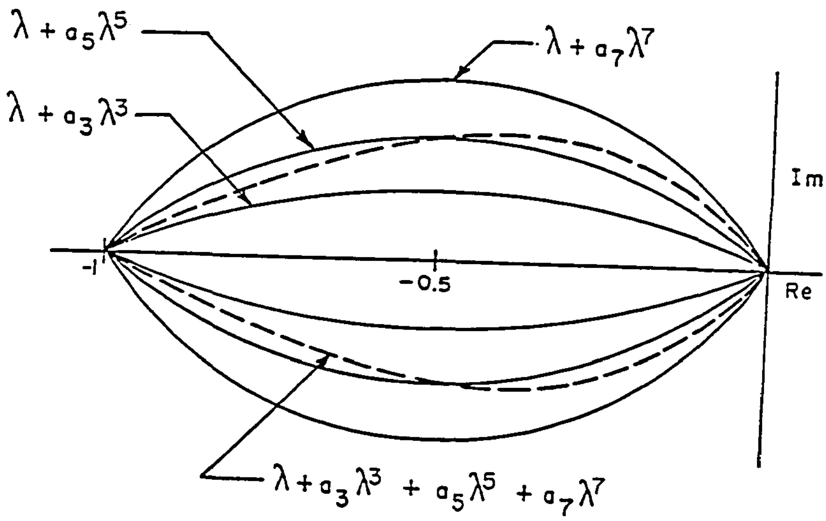


Figure 4. Envelopes of different nonlinearities

that for higher order nonlinearity, it was difficult to isolate the complex roots for finding the jump-to points.

In 1977, Prusty (30) developed an equation for a polynomial type nonlinearity that he used to evaluate the threshold frequency for a quintic nonlinearity. However, he stated that it was difficult to perform the same analysis on higher order nonlinearity.

Accurate methods of representation were also developed by Teape, Slater, Simpson, and Wood (29). However, the amount of computing time that has to be spent in simulations can be very great.

In 1978, Feldman and Cappabianca (33) performed ferroresonance studies with straight line representation of the transformer characteristic as shown in Figure 3d. It will be seen from this figure that allowance is made for the iron losses which is an improvement over previous studies based on straight line representation. However, previous studies indicate that of the three core magnetization phenomena -- saturation, hysteresis, and eddy currents -- only saturation has a significant effect on ferroresonance studies (26,30).

In 1979, Mukherjee and Ray (34) developed a model to determine the switching transients in a series circuit by assuming its magnetic characteristic as piecewise linear.

In the power industry, transformers could be produced with different magnetic material composition and different sizes ranging from 0.015 MVA to over 30 MVA. Therefore, it would be unrealistic to represent all these transformers with the same single nonlinear quintic term of G. W. Swift. Therefore, Swift's representation of a magnetization curve by a single-nonlinear quintic term has limited the application to a specific type of transformer.

In addition, Karlicek and Taylor (35) had reported the occurrence of ferroresonance in the instrument potential transformer circuit of the power system. This is not a power transformer, so Swift's model would not apply. Therefore, if need arises to represent the magnetization curve in different circuits with higher order single term non-linearity or the general lambda polynomial type, it would not be possible to obtain the jump point lambdas analytically.

However, if the transformer is replaced by a Describing Function based on two slopes of the magnetization curve, then it will be possible to obtain jump point lambdas or currents analyitically for power transformers as well as the instrument potential transformers.

The method of the Incremental Describing Function applied to the two-slope representation of a typical magnetic core characteristic is preferable over the direct two-slope linearization in particular (21) and piecewise approximation in general (34) because it does not depend on a trial and error procedure and the system of differential equations for loaded or unloaded circuits can be easily transformed into algebraic equations. Therefore, higher order systems resulting from practical applications can also be treated with nearly the same facility.

Even though progress has been made with the studies mentioned so far, the author felt that an approach is needed to study the phenomenon in a power system network where the effort in obtaining a solution is not dependent on the complexity of the circuit.

III. DESCRIPTION OF METHODS USED TO STUDY FERRORESONANCE

Using a pi circuit example, a short mathematical derivation is developed for each of the following analytical techniques, applied to ferroresonance occurring in power circuits.

A. Piecewise Linearization Technique

The approximation for the magnetization characteristic of the transformer in unloaded pi circuit is done in the vicinity of operating points and the actual curve is approximated by small straight line segments as illustrated in Figure 5. In order to maintain a reasonable accuracy, a small signal constraint must be imposed around these points. Total solution is obtained by combining these segmental solutions. The system differential equations are then solved on a sectionalized basis by matching boundary conditions each time a change occurs from one linear region to another. Such boundary or initial conditions matching is extremely tedious. Therefore, the amount of work increases and the solution becomes much more complicated as the number of segments increase, especially for higher order differential equation models.

To illustrate this analytical technique, the pi circuit given in Figure 6 is used.

The equations that apply to the pi circuit of Figure 6 are developed in Chapter 10 and summary is shown below:

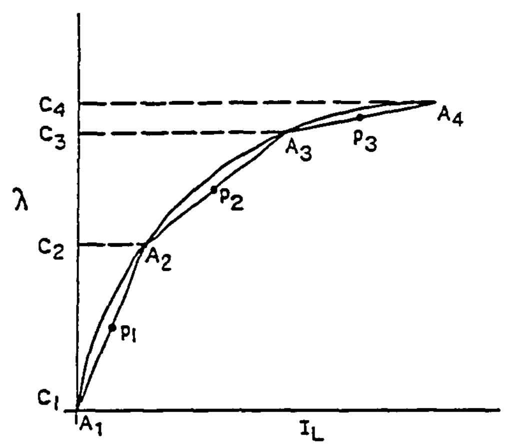


Figure 5. Piecewise linearization of the magnetization curve

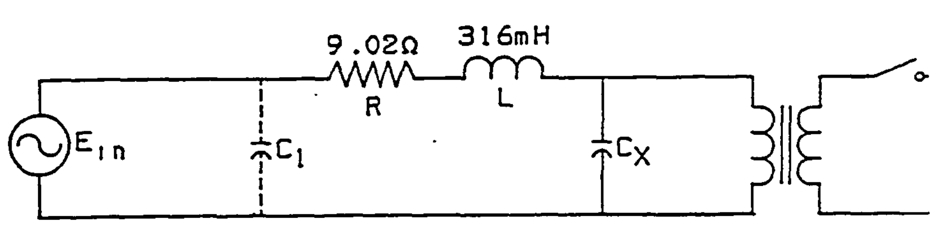


Figure 6. Pi circuit used in illustrating piecewise linearization analysis technique

$$RI + L\frac{dI}{dt} + \frac{d\lambda}{dt} = E_m \sin(\omega t+\phi) \tag{10.1}$$

$$\frac{d\lambda}{dt} = \frac{1}{c}\int I_c dt = \frac{Q_c}{c} \tag{10.2}$$

$$\lambda = b_k(I_L-a_k) + c_k \tag{10.3}$$

$$I = I_L + I_c \tag{10.4}$$

From the above four equations, the following third order differential equation is obtained. Since

$$\frac{d\lambda}{dI_L} = \frac{d\lambda}{dt}\quad\frac{dt}{dI_L} = b_k$$

$$\therefore \frac{d\lambda}{dt}\quad\frac{dt}{dI_L} = b_k = L_k \quad \text{and} \quad \frac{d\lambda}{dt} = L_k\frac{dI_L}{dt} \tag{10.5}$$

and

$$\frac{d\lambda}{dt} = \frac{Q_c}{c}$$

$$\therefore \quad L_k\frac{dI_L}{dt} = \frac{Q_c}{c} \quad \text{and} \quad \frac{dI_L}{dt} = \frac{Q_c}{L_k c}$$

since

$$I = I_L + I_c \qquad \text{and} \qquad c\frac{d\lambda}{dt} = \int I_c dt$$

$$I = I_L + \frac{dQ_c}{dt} \qquad cL_k \frac{dI_L}{dt} = \int I_c dt = Q_c$$

$$cL_k \frac{d^2 I_L}{dt^2} = I_c = \frac{dQ_c}{dt}$$

$$\therefore \; I = I_L + cL_k \frac{d^2 I_L}{dt^2} \qquad cL_k \frac{d^3 I_L}{dt^3} = \frac{dI_c}{dt}$$

$$RI + L\frac{dI}{dt} + L_k \frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

$$RI + L\frac{dI}{dt} + \frac{Q_c}{c} = E_m \sin(\omega t+\phi)$$

$$CRI + LC\frac{dI}{dt} + Q_c = E_m C \sin(\omega t+\phi)$$

$$Q_c = E_m C \sin(\omega t+\phi) - CRI - LC\frac{dI}{dt} \tag{10.6}$$

$$RI + L\frac{dI}{dt} + \frac{d\lambda}{dt} = E_m \sin(\omega t+\phi)$$

$$R(I_L+I_c) + L\frac{d}{dt}(I_L+I_c) + L_k \frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

$$RI_L+RI_c + L\frac{dI_L}{dt} + L\frac{dI_c}{dt} + L_k \frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

$$RI_L+RI_c + (L+L_k)\frac{dI_L}{dt} + L\frac{dI_c}{dt} = E_m \sin(\omega t+\phi)$$

$$RI_L + R(CL_k \frac{d^2I_L}{dt^2}) + (L+L_k)\frac{dI_L}{dt} + L(CL_k)\frac{d^3I_L}{dt^3} = E_m\sin(\omega t+\phi)$$

$$LCL_k \frac{d^3I_L}{dt^3} + RCL_k \frac{d^2I_L}{dt^2} + (L+L_k)\frac{dI_L}{dt} = E_m\sin(\omega t+\phi)$$

or

$$\frac{d^3I_L}{dt^3} + \frac{RCL_k}{LCL_k}\frac{d^2I_L}{dt^2} + (\frac{L+L_k}{LCL_k})\frac{dI_L}{dt} + \frac{RI_L}{LCL_k} = \frac{E_m}{LCL_k}\sin(\omega t+\phi) \quad (10.7)$$

However, a higher than third order system would have resulted if we had started with a general rather than a pi circuit.

Characteristic equation:

$$m^3 + \frac{R}{L}m^2 + \frac{L+L_k}{LCL_k}m + \frac{R}{LCL_k} = 0$$

Roots may be found by the numerical analysis technique called the secant method. However, since the last term is less significant, compared to the remaining terms, we will neglect it to simplify the solution.

$$m^3 + \frac{R}{L}m^2 + \frac{L+L_k}{LCL_k}m = 0$$

$$m(m^2 + \frac{R}{L}m + \frac{L+L_k}{LCL_k}) = 0$$

$$m_1 = 0$$

$$m_{2,3} = -\frac{R}{2L} \mp \frac{1}{2}\sqrt{(\frac{R}{L})^2 - 4(\frac{L+L_k}{LCL_k})}$$

$$= -\frac{R}{2L} \mp \sqrt{\frac{R^2}{4L^2} - \frac{4(L+L_k)}{4(LCL_k)}}$$

where $L_{eq} = \frac{LL_k}{L+L_k}$

and

$$m_{2,3} = -\frac{R}{2L} \mp \sqrt{\frac{R^2}{4L^2} - \frac{1}{L_{eq}C}}$$

$$= -\frac{R}{2L} \mp j\sqrt{\frac{1}{L_{eq}C} - \frac{R^2}{4L^2}}$$

$$m_{2,3} = -\frac{R}{2L} \mp j\nu$$

$$\therefore\ I_{compl.} = Ae^{ot} + Be^{-\frac{R}{2L}t} \sin(\nu t+\theta_2)$$

or

$$I_{compl.} = A + Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2)$$

$$I_{particular_1} = \frac{\frac{E_m}{LCL_k} \cdot e^{j\omega(t+\frac{\phi}{\omega})}}{(+j\omega)^3 + \frac{R}{L}(j\omega)^2 + \frac{L+L_k}{LL_kC}(j\omega)}$$

Further simplification of this equation with $L_{eq} = \frac{LL_k}{L+L_k}$ yields

$$I_{p_1} = \frac{E_m}{LL_k\omega^2 Z} e^{j(\omega t+\phi+\theta_1)}$$

Since the source function is sinusoidal:

$$I_p = \frac{E_m}{LL_k\omega^2 Z} \sin(\omega t+\phi+\theta_1) = \frac{E_m}{XX_k Z}\sin(\omega t+\phi+\theta_1)$$

where

$$\theta_1 = \tan^{-1} \frac{\frac{1}{L_{eq}\omega} - C\omega}{\frac{CR}{L}}$$

$$Z = \sqrt{(\frac{CR}{L})^2 + (\frac{1}{L_{eq}\omega} - c\omega)^2}$$

An analytical solution for the first two segments using this technique is given in Appendix B. A summary of these results are given below:

1. Solution of the above equation

$$I_L = A + B\ e^{-\frac{R}{2L}t} \sin(\nu t+\theta_2) + \frac{E_m}{LL_k\omega^2 Z} \sin(\omega t+\phi+\theta_1)$$

$$V_L = \frac{d\lambda}{dt} = L_k \frac{dI_L}{dt}$$

Steady state solution:

$$V_{Lss} = \frac{E_m}{L\omega Z} \cos(\omega t+\phi+\theta_1)$$

2. The transformer voltage equation for the first section is:

$$E_{t_1} = \frac{E_m}{L\omega Z} \qquad 0 \leq t \leq t_1$$

$$\text{where } Z = \sqrt{\left(\frac{CR}{L}\right)^2 + \left(\frac{1}{L_{eq}\omega} - C\omega\right)^2}$$

$$L_{eq} = \frac{L_1 L}{L_1 + L}$$

3. Evaluation of time t at $I = A_2$

The Newton-Raphson or any similar numerical analysis method is used to find the value of t at which the current function crosses the limit $I = A_2$. In the following equation a guess is made as to the time t' corresponding to that value of current at $I = A_2$:

$$\Delta t = \frac{A_2 - F(t')}{\dot{F}(t')}$$

$$\text{where } \dot{F}(t) = \frac{dF(t)}{dt} = \frac{dI_L}{dt}$$

and

$$t_1 = \Delta t + t'$$

iterations until $\Delta t \rightarrow 0$

and t is known.

4. The transformer voltage equation for the second section is:

$$V_{Lss} = \frac{d\lambda}{dt} = \left(\frac{E_m}{LZL_1\omega} - \frac{C\,E_m\omega}{LZ}\right)\cos(\omega t_1+\varphi+\varphi_2)$$

$$t_1 < t \leq t_2$$

For the 60 Hz point the amplitude of the cosine is a function of R, L, C, and E_m. If we were to study ferroresonance with this equation we should search for values of R, L, C, and E_m that would increase the magnitude of this amplitude above a certain known nominal transformer voltage.

Therefore, using piecewise linearization to study ferroresonance requires an exhaustive process of blind trial and error of a large number of values for R, L, C, and E_m to locate the critical values for the production of ferroresonance.

In order to be able to make a reasonable comparison of the results with those of the present analysis and the method of G. W. Swift, the following parameters have to be used:

R = 0.002 P.U.

L = 0.025 P.U.

C = 50 P.U.

E_m = 0.22 P.U.

$$Z = \sqrt{(\frac{CR}{L})^2 + (\frac{1}{L_{eq}\omega} - C\omega)^2} \qquad (10.9)$$

$$L_k \frac{dI_L}{dt} = \frac{d\lambda}{dt} = \frac{E_m}{L\omega} [\frac{1}{\sqrt{(\frac{CR}{L\omega})^2 + (\frac{1}{L_{eq}\omega} - C\omega)^2}}] \qquad (10.10)$$

and

$$\lambda = \frac{E_m}{L} [\frac{1}{\sqrt{(\frac{CR}{L})^2 + (\frac{1}{L_{eq}} - C)^2}}] \text{ P.U.}$$

since

$$L_{eq} = \frac{LL_k}{L+L_k}$$

$$L_{eq} = \frac{LL_1}{L+L_1}$$

$$= \frac{0.025}{1.025}$$

$$= 0.0243902 \text{ P.U.}$$

$$L_1 = 1$$

$$L_2 = \frac{1}{27.434377}$$

$$= 0.0364241 \text{ P.U.}$$

$$\lambda_1 = \frac{0.22}{0.025}[\frac{1}{\sqrt{(\frac{50x0.002}{0.025})^2 + (\frac{1}{0.0243902} - 50)^2}}]$$

$$= \frac{0.22}{0.025}[0.1015346]$$

$$\lambda_1 = 0.8935 \text{ P.U.}$$

$$\frac{d\lambda}{dt} = L_2 \frac{dI_L}{dt} = (\frac{E_m}{LZ} - \frac{CE_m}{LZ})L_2$$

or

$$\lambda = \frac{E_m}{LZ}(1-C)L_2 \text{ P.U.}$$

$$= \frac{0.22}{0.025}\left[\frac{49}{\sqrt{\{\frac{(50)(0.002)}{0.025}\}^2 + (\frac{1}{0.0243902} - 50)^2}}\right](0.0364241)$$

$$= (8.8)(0.1015346)(49)(0.0364241)$$

$$= 1.5947097 \text{ P.U.}$$

$$\simeq 1.595 \text{ P.U.}$$

5. In summary, the results obtained are shown below:

Method	The voltage (P.U.) Jump from	Jump to	Circuit's critical input voltage (P.U.)
Piecewise linear method	0.89	1.595	0.22
Present analysis	0.94	1.46	0.22
Experiment	1.08	1.44	0.17

Values obtained by the proposed methodology are closer to the experimental values than those obtained by two segments of the linear approximation (see data).

B. G. W. Swift's Method

In recent years, the emphasis has been shifted to the use of the Describing Function, developed from the nonlinear control theory. The Describing Function is a complex gain which at fundamental frequency (60 Hertz) modifies the amplitude and phase angle of the input to the nonlinearity:

$$N(\lambda_m, \omega) = \frac{\text{Phasor representation of output component at frequency } \omega}{\text{Phasor representation of input at frequency } \omega}$$

Unlike piecewise linearization, the describing function exhibits dependence on input signal amplitude, the basic characteristic of nonlinear behavior. For instance, a small signal linearization about the origin would approximate the nonlinear function by a fixed gain, the slope of the nonlinear function at the origin. If the signal at the input to the nonlinearity extends into the saturation region, then it seems obvious that the effective gain of the nonlinearity $N(\lambda_m)$ is lower than that for a small signal around the origin. Therefore, small signal linearization does not exhibit the desirable dependence on input signal amplitude.

A special case of the two sinusoidal Input Describing Function (TSIDF) is that of an amplitude of one input sinusoid being much smaller than the amplitude of the other. Under this circumstance, a simple closed-form solution exists for the nonlinearity gain to the small amplitude input component; it is called the Incremental Input Describ-

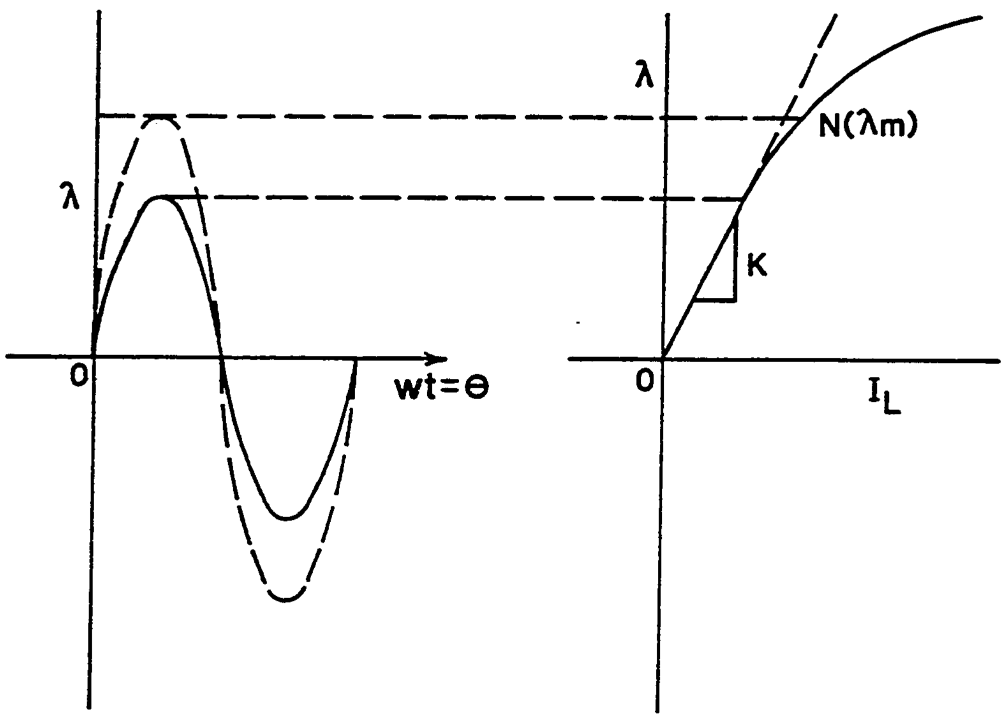


Figure 7. Small signal linearization vs. Describing Function approximation

ing Function. This function is of considerable interest to study ferroresonance:

$$K(\lambda_m,\omega) = \frac{\text{Fundamental of incremental output phasor}}{\text{Incremental input phasor}}$$

For the unloaded pi circuit, assume that the input to the Describing Function for the nonlinearity $N(\lambda_m)$ is given by:

$$\lambda(t) = \lambda_m \cos(\omega t+\phi) + \mu \cos \omega t \quad \mu << \lambda_m \qquad (10.14)$$

and μ is an incremental perturbation amplitude in $\lambda(t)$, ϕ is any phase relationship between the main signal and the perturbation signal equal to $\mu \cos \omega t$.

Assume that the input by passing through the nonlinearity $N(\lambda_m)$ is multiplied by a gain factor $K(\lambda_m,\phi)$ which is the fundamental component transfer function gain that depends on λ_m and ϕ.

In this situation, $K(\lambda_m,\phi)G(jw) + 1 = 0$ will be the characteristic equation, provided the solution harmonics are filtered sufficiently by the $G(jw)$ transfer function so that the typical linear system stability criterion can be used. Thus, the stability criterion is given by

$$K(\lambda_m,\phi)G(jw) = -1$$

or $G(jw) = \frac{1}{K(\lambda_m,\phi)}$ where the intersection points of the LHS and RHS of this equation are the critical points.

In order to be able to plot G(jw) and $K(\lambda_m,\phi)$ on the same graph and find these critical points, we must first solve for $K(\lambda_m,\phi)$. This is accomplished by using the diagram in Figure 8.

Figure 8. Diagram for $K(\lambda_m,\phi)$ determination

Using Swift's model given by $i_L = \lambda+4\lambda^5$, we find that:

$$i_L = [\lambda_m \cos(\omega t+\phi) + \mu \cos \omega t] + 4[\lambda_m \cos(\omega t+\phi) + \mu \cos \omega t]^5 \qquad (10.16)$$

Reducing this function by omitting terms in μ^2 or higher powers of μ because $\mu << \lambda_m$ and omitting terms in frequencies of 2ω or greater because G(jw) is essentially a low pass filter we obtain:

$$i_L = \lambda_m \cos(\omega t+\phi) + \mu \cos \omega t + \frac{5}{2}\lambda_m^5 \cos(\omega t+\phi)$$

$$+ \frac{15}{2}\lambda_m^4 \mu \cos \omega t + 5\lambda_m^4 \mu \cos(\omega t+2\phi)$$

The full development of this solution for i_L is given in Appendix B.

From the definition of the incremental sinusoidal input Describing Function $K(\lambda_m,\phi)$ given by:

$$K(\lambda_m,\phi) \triangleq \frac{\text{Fundamental of incremental output phasor}}{\text{Incremental input phasor}}$$

we get

$$K(\lambda_m,\phi) = \frac{\mu+\frac{15}{2}\lambda_m^4 \mu + 5\lambda_m^4 \mu e^{j2\phi}}{\mu}$$

or

$$K(\lambda_m,\phi) = 1 + \frac{15}{2}\lambda_m^4 + 5\lambda_m^4 e^{j2\phi} \qquad (10.17)$$

If we let

$$A = 1+\frac{15}{2}\lambda_m^4 \quad \text{and} \quad B = 5\lambda_m^4$$

then

$$K = A+Be^{j2\phi}$$

In order to graph $(-\frac{1}{K})$

where $K = A+Be^{j2\phi}$

and $K-A = Be^{j2\phi}$

or $\frac{K-A}{B} = e^{j2\phi}$

we get

$$\frac{B}{K-A} = e^{-j2\phi} = M$$

$$KM-AM = B$$

so

$$K = \frac{B+AM}{M}$$

and

$$\frac{1}{K} = \frac{M}{B+AM} = \frac{e^{-j2\phi}}{B+Ae^{-j2\phi}} = \frac{1}{A+Be^{+j2\phi}}$$

which yields

$$\frac{1}{K} = \frac{1}{B+Ae^{-j2\phi}} \quad \text{and} \quad \left|\frac{1}{K}\right| = \left|\frac{1}{B+Ae^{-j2\phi}}\right| = \left|\frac{1}{B+Ae^{+j2\phi}}\right|$$

or

$$\frac{1}{K} = \left|\frac{1}{B+Ae^{-j2\phi}}\right| e^{-j2\phi}$$

Using the fact that $\frac{K-A}{B} = e^{j2\phi}$, we can write the expression

$$\frac{e^{-j2\phi}}{B+A(\frac{K-A}{B})} = \frac{e^{-j2\phi}}{B+\frac{AK-A^2}{B}}$$

or

$$= \frac{e^{-j2\phi}}{\frac{B^2+AK-A^2}{B}}$$

so that

$$\frac{1}{K} = \frac{Be^{-j2\phi}}{B^2+AK-A^2}$$

Rearrangement gives

$$B^2+AK-A^2 = KBe^{-j2\phi}$$

$$B^2-A^2 = -AK + KBe^{-j2\phi}$$

$$B^2-A^2 = -K(A-Be^{-j2\phi})$$

and

$$-\frac{1}{K} = \frac{A-Be^{-j2\phi}}{B^2-A^2} = \frac{A}{B^2-A^2} - \frac{B}{B^2-A^2}e^{-j2\phi} \qquad (10.18)$$

This is the equation for a family of circles with a radius $\frac{B}{B^2-A^2}$ and a center at $(-\frac{A}{B^2-A^2},0)$.

For each value of λ_m between 0.3 to 1.4 P.U., the corresponding circle center and radius is calculated and plotted on a complex plane. The result is the family of circles of Figure 9 with its centers located on the negative real axis and moving to the right toward the origin as the constant λ_m value increases. If two smooth curves are drawn starting from the origin and touching all these circles from above and below the negative real axis, the resulting curve is the envelope of the stability curve that encloses the ferroresonant region.

Flux Linkage	$A = 1+\frac{15}{2}\lambda_m^4$	$B = 5\lambda_m^4$	Circle Center	Circle Radius
λ_m(p.u.)	A(p.u.)	B(p.u.)	$\frac{A}{B^2-A^2}$	$\frac{B}{B^2-A^2}$
1.4	29.812	19.208	-0.057	-0.037
1.2	16.552	10.368	-0.0994	-0.0623
1.0	8.500	5.000	-0.1799	-0.106
0.9	5.921	3.281	-0.2437	-0.1353
0.8	4.072	2.048	-0.3287	-0.1653
0.7	2.801	1.200	-0.4373	-0.1874
0.6	1.972	0.648	-0.5685	-0.1868
0.5	1.469	0.3125	-0.713	-0.1517
0.4	1.192	0.128	-0.849	-0.091
0.3	1.061	0.0405	-0.944	-0.036

For the linear part of the pi circuit, the component values used in this example are given by:

R = 0.002 P.U.

L = 0.021 P.U. or 0.025 P.U.

C = 50 P.U.

where $G_p(s)$ is derived in Chapter IV.

$$G_p(s) = \frac{R+SL}{S(S^2LC+SRC+1)} \qquad (4.18)$$

or

$$G_p(s) = \frac{0.002+S(0.021)}{S[S^2(0.021)(50)+S(0.002)(50)+1]}$$

$$= \frac{0.002+0.021S}{S[1.05S^2+0.1S+1]}$$

Now letting $S=j\omega$ for steady state sinusoidal analysis we get:

$$= \frac{0.002+0.021j\omega}{j\omega[1.05(j\omega)^2+0.1(j\omega)+1]}$$

$$= \frac{0.002+0.021j\omega}{j\omega[-1.05\omega^2+0.1\omega j+1]}$$

$$= \frac{0.002+0.021j\omega}{-1.05\omega^3 j-0.1\omega^2+j\omega}$$

$$= \frac{0.002+0.021j\omega}{-0.1\omega^2+j(\omega-1.05\omega^3)}$$

$$= \frac{(0.002+0.021j\omega)(-0.1\omega^2-j(\omega-1.05\omega^3)]}{[-0.1\omega^2+j(\omega-1.05\omega^3)][-0.1\omega^2-j(\omega-1.05\omega^3]}$$

$$G_p(j\omega) = \frac{-0.0002\omega^2-j0.002(\omega-1.05\omega^3)-j0.0021\omega^3+0.021\omega(\omega-1.05\omega^3)}{0.01\omega^4+(\omega-1.05\omega^3)^2}$$

$$G_p(j\omega) = \frac{-0.0002\omega^2 - j0.002\omega + j0.0021\omega^3 - j0.0021\omega^3 + 0.021\omega^2 - 0.02205\omega^4}{0.01\omega^4 + (\omega - 1.05\omega^3)^2}$$

$$= \frac{0.0208\omega^2 - 0.02205\omega^4 - j0.002\omega}{0.01\omega^4 + (\omega - 1.05\omega^3)^2}$$

$$G_p(j\omega) = \frac{0.0208\omega^2 - 0.02205\omega^4}{0.01\omega^4 + (\omega - 1.05\omega^3)^2} - \frac{j0.002\omega}{0.01\omega^4 + (\omega - 1.05\omega^3)^2}$$

$$Re[G(j\omega)] = \frac{0.0208\omega^2 - 0.02205\omega^4}{0.01\omega^4 + (\omega - 1.05\omega^3)^2} = \frac{0.0208 - 0.02205\omega^2}{0.01\omega^2 + (1 - 1.05\omega^2)^2}$$

$$I_{mag}[G(j\omega)] = \frac{0.002\omega}{0.01\omega^4 + (\omega - 1.05\omega^3)^2} = \frac{-0.002\omega}{0.01\omega^3 + \omega(1 - 1.05\omega^2)^2}$$

Data for two values of L are given below:

G(jω) for L = 0.021 P.U.:

ω	ReG(jω)	ImG(jω)	ω	ReG(jω)	ImG(jω)
0	0.0208	−∞	0.28	0.0226252	−0.0084739
0.02	0.0208085	−0.1000837	0.3	0.0229225	−0.0081219
0.04	0.0208343	−0.0501676	0.32	0.023249	−0.0078366
0.06	0.0208774	−0.0335856	0.34	0.0236067	−0.0076085
0.08	0.020938	−0.0253378	0.36	0.0239982	−0.0074307
0.1	0.0210164	−0.0204246	0.4	0.0248939	−0.0072065
0.12	0.0211131	−0.0171798	0.5	0.0279783	−0.0073206
0.14	0.0212284	−0.0148893	0.6	0.0329385	−0.0085364
0.16	0.0213631	−0.0131965	0.7	0.0415423	−0.0118746
0.18	0.0215178	−0.0119034	0.8	0.0586749	−0.0219329
0.2	0.0216933	−0.0108913	0.9	0.0965345	−0.0729788
0.22	0.0221108	−0.010085	0.92	0.1025018	−0.1042782
0.24	0.0221108	−0.0094346	0.94	0.0936981	−0.1514162
0.26	0.0223552	−0.0089057	0.96	0.0466562	−0.2030429

G(j) for L = 0.021 P.U.

ω	ReG(jω)	ImG(jω)	ω	ReG(jω)	ImG(jω)
0.97	$5.23\times10^{-8}\simeq0$	-0.216218	4.0	-0.0013291	-0.000002
0.99	-0.0761831	-0.1897242	5.0	-0.0008317	-0.0000006
1.0	-0.1	-0.16	6.0	-0.0005705	-0.0000002
2.0	-0.0065564	-0.0000973	7.0	-0.0004163	-0.0000001
3.0	-0.0024849	-0.0000093	1000	$-2\times10^{-8}\simeq0$	$-1.8\times10^{-18}\simeq0$

G(jω) for L = 0.025 P.U.

ω	ReG(jω)	ImG(jω)	ω	ReG(jω)	ImG(jω)
0	0.0248	$-\infty$	0.88	0.0684307	-0.2592071
0.02	0.0248122	-0.1000997	0.89	0.0058458	-0.2802503
0.04	0.0248489	-0.0501998	0.89	$-1.4\times10^{-8}\simeq0$	-0.2806333
0.06	0.0249103	-0.0336341	0.92	-0.1394995	-0.1837938
0.08	0.0249967	-0.0254032	0.94	-0.14236	-0.1076955
0.1	0.0251088	-0.0205074	0.96	-0.1237624	-0.0644596
0.12	0.0252471	-0.0172807	1.0	-0.0889655	-0.0275862
0.14	0.0254124	-0.0150092	1.02	-0.0765856	-0.019470
0.16	0.024606	-0.0133365	1.04	-0.0668052	-0.014274
0.18	0.0258289	-0.0120646	1.06	-0.058977	-0.0107905
0.2	0.0260826	-0.0110754	1.08	-0.052613	-0.0083632
0.22	0.0263689	-0.0102938	1.10	-0.0473602	-0.0066174
0.24	0.0266896	-0.0096701	1.2	-0.030868	-0.0025469
0.26	0.027047	-0.0091704	1.3	-0.0223286	-0.0012263
0.28	0.0274439	-0.0087708	1.4	-0.0171764	-0.0006732
0.30	0.0278832	-0.0084543	1.5	-0.0137597	-0.0004031
0.32	0.0283685	-0.0082085	1.6	-0.011345	-0.0002569
0.34	0.0289038	-0.0080246	1.7	-0.0095582	-0.0001716
0.36	0.0294937	-0.0078966	1.8	-0.0081897	-0.000119
0.38	0.0301438	-0.0078202	1.9	-0.0071128	-0.0000851
0.4	0.0308603	-0.007783	2.0	-0.0062469	-0.0000623
0.5	0.0357514	-0.0084183	3.0	-0.0024388	-0.0000063
0.6	0.0442666	-0.0108897	4.0	-0.0013158	-0.0000014
0.7	0.0611875	-0.0184265	5.0	-.0008264	-0.0000004
0.8	0.1034483	-0.0538793	6.0	-0.0005682	-0.0000002
0.82	0.117755	-0.0758303	7.0	-0.0004149	$-7.870\times10^{-8}\simeq0$
0.84	0.1310772	-0.1134868	10.0	$-2\times10^{-8}\simeq0$	$-1.28\times10^{-8}\simeq0$
0.86	0.1288537	-9.1775762			

G. W. Swift's (26) representation of the magnetization curve by one nonlinear quintic term limited the application to a specific type of transformers. According to Ashok Kumar et al. (2), modern power transformers with fast saturation type characteristic, which are represented by the seventh order, will make the system more susceptible to ferroresonance. Therefore, the objection reported in the literature to the use of one nonlinear term of G. W. Swift is that the prediction of susceptibility of various practical circuits to ferroresonance obtained by the use of a single nonlinear term analysis tends to give pessimistic results. The data below show a comparison of results obtained by the two-slope Describing Function model proposed in this study versus those obtained by using Swift's linear plus fifth order term approximation and his experimental results. Note how close the results of the present two slope Describing Function core model are to those obtained by Swift as shown in Figure 10 and the comparison data. Critical lambdas in the stability curve of Figure 10 as compared to those published by G. W. Swift (26) are for the critical values of: (1) R = 0.002 P.U., (2) L = 0.025 P.U., and (3) C = 50 P.U.

Method	TRF voltage (p.u.) Jump from	TRF voltage (p.u.) Jump to	Circuit's critical input voltage (p.u.)
G. W. Swift (Analytical)	0.96	1.38	0.19
Present analysis (Analytical)	0.94	1.46	0.22
Experimental (G. W. Swift)	1.08	1.44	0.17

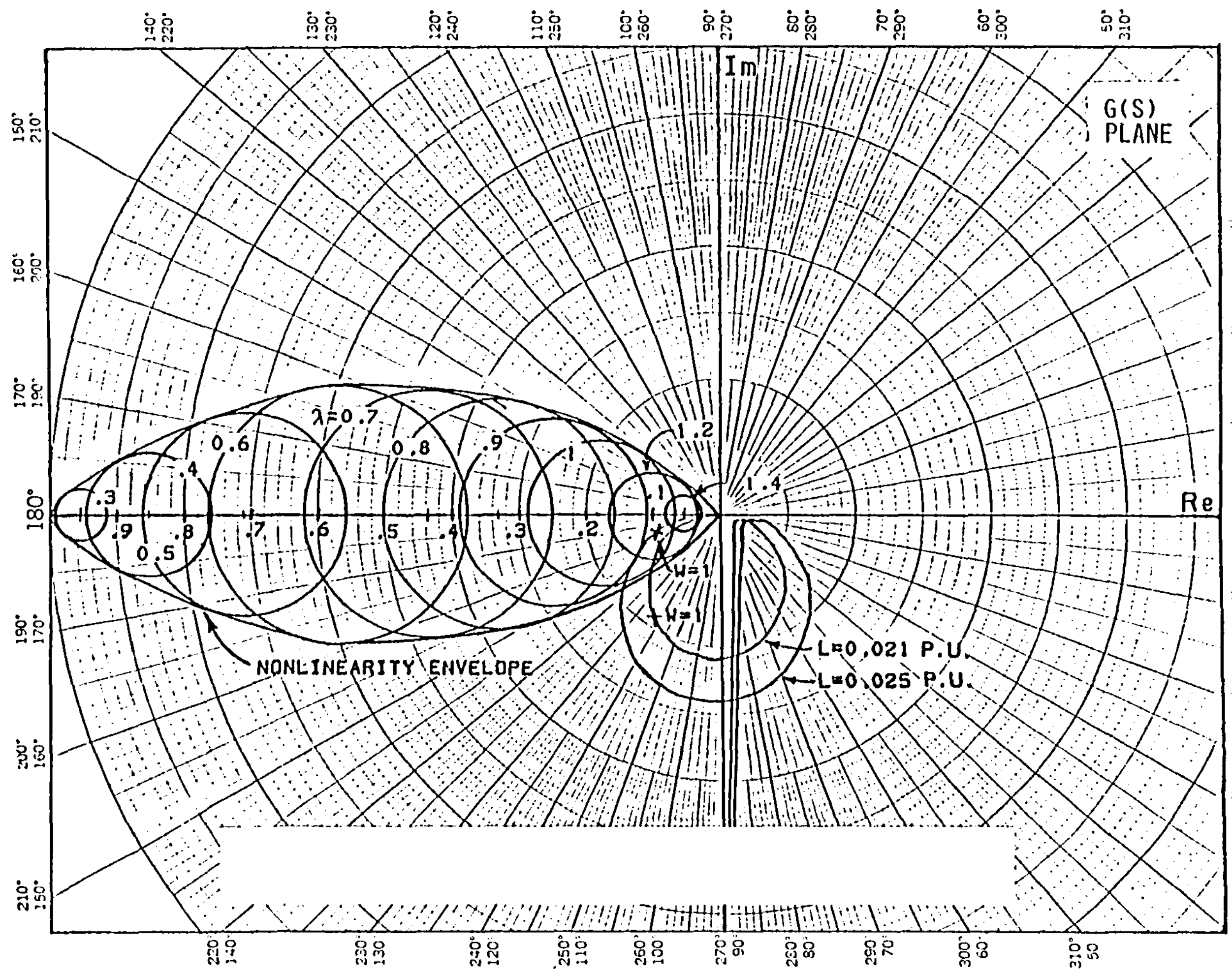


Figure 9. Stability and frequency response curves of the pi circuit used by G. W. Swift

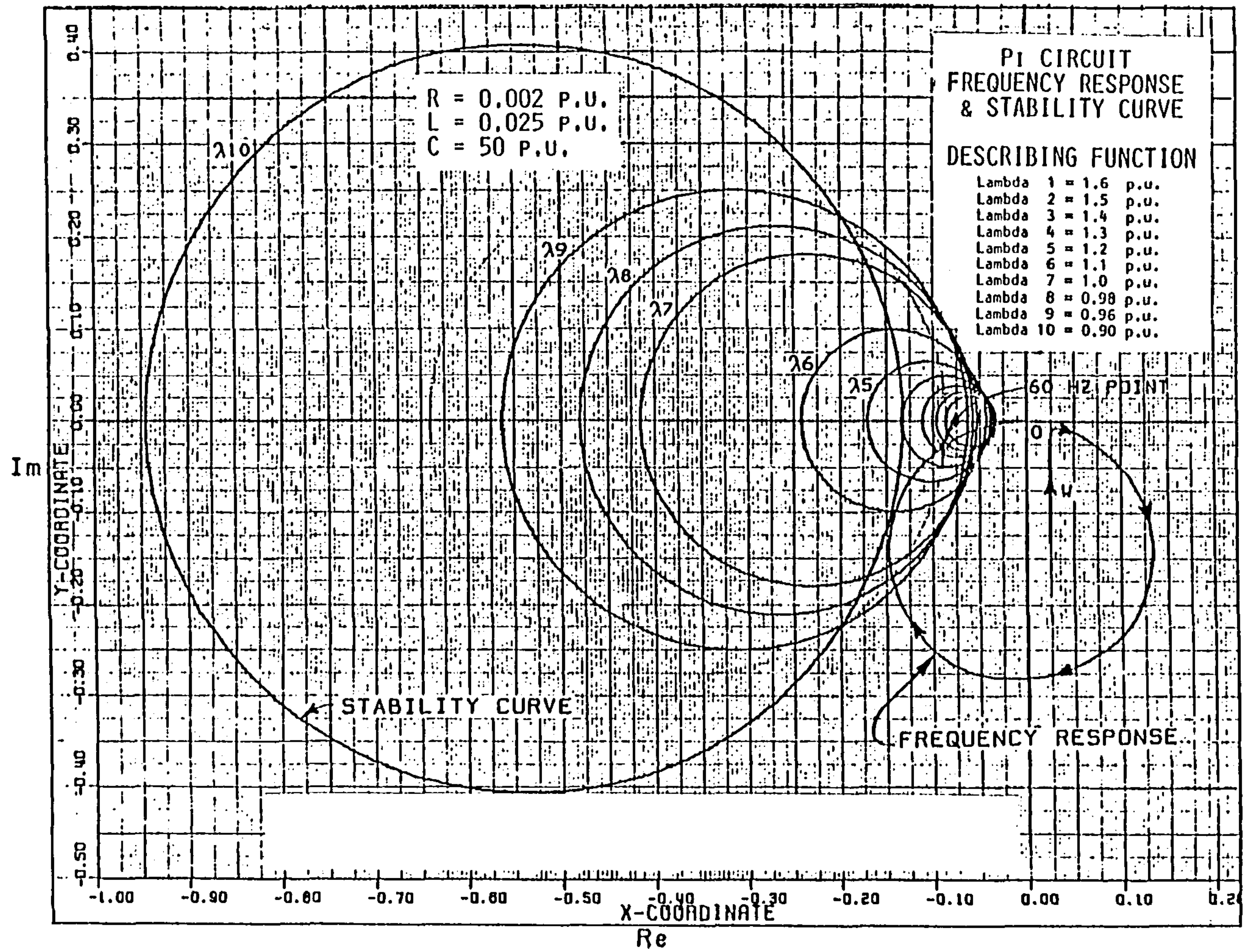


Figure 10. Stability and frequency response curves for values used by G. W. Swift

Using G. W. Swift's method, transformer voltage switching and transient times as defined in Chapter V were also calculated. A comparison of results indicates that although transformer voltage switching times are in agreement, their transient times are not. Transient time obtained by G. W. Swift's method appears to be too long when compared with the experimental results of about 70 ms. Switching time is the time required to reach the critical λ_m and transient time is the time required to change from linear mode of critical λ_m to that of the nonlinear mode of λ_m.

Method	Transformer voltage Switching time	Transient time
Swift	79.3 s	127.2 s
Present analysis	79.26 s	67.3 ms
Experimental	70 s	70 ms

Summary of results obtained by the three methods are shown in the data below:

Method	TRF voltage (P.U.) Jump from	Jump to	Circuit's critical input voltage (P.U.)
G. W. Swift's	0.96	1.38	0.19
Piecewise linear	0.89	1.595	0.22
Present analysis	0.94	1.46	0.22
Experiment	1.08	1.44	0.17

C. Gear's Digital Program

In this study, Gear's digital computer program is applied to the analysis of ferroresonance. This program consists of a package of subroutines known as the Gear package developed by C. W. Gear to solve Systems of Ordinary Differential Equations. The method is based on the numerical analysis algorithm of variable step type which adjusts the integration interval to the proper size depending on function behavior. Automatic control of step size and order is performed by the package for efficiency in the integration. The program's user must provide a subroutine describing the particular problem and a main program that calls subroutine DRIVE in the package. Description of the requirements and results of the program are:

1. Program's input

a. State variables of system To solve the equations

$$e_{in} = Ri + L\frac{di}{dt} + \frac{d\lambda}{dt} ,$$

$$\frac{1}{c}\int i_c dt = \frac{d\lambda}{dt} ,$$

$$i = i_c + i_T ,$$

and $i_T = \lambda+4\lambda^5$ for the pi circuit of Figure 11.

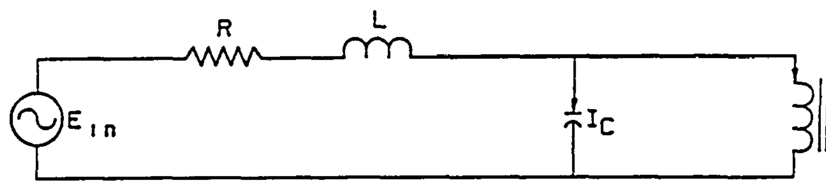


Figure 11. Pi circuit used in the analysis

The resulting third order differential equation must first be converted into three first order differential equations as illustrated in the state variable formulation given below. The basic equations are:

$$e_{in} = Ri + L\frac{di}{dt} + \frac{d\lambda}{dt}$$

and

$$i = i_c+i_T$$

Combining these equations with those given above, we get:

$$e_{in} = R(i_c+i_T) + L\frac{d}{dt}(i_c+i_T) + \frac{d\lambda}{dt}$$

$$e_{in} = Ri_c + Ri_T + L\frac{di_c}{dt} + L\frac{di_T}{dt} + \frac{d\lambda}{dt}$$

Substituting for i_T in the above equation

$$e_{in} = Ri_c + R(\lambda+4\lambda^5) + Lc\frac{d^3\lambda}{dt^3} + L\frac{d}{dt}(\lambda+4\lambda^5) + \frac{d\lambda}{dt}$$

which gives

$$e_{in} = RC\frac{d^2\lambda}{dt^2} + R(\lambda+4\lambda^5) + LC\frac{d^3\lambda}{dt^3} + L\frac{d\lambda}{dt} + 20\lambda^4 \frac{d\lambda}{dt} + \frac{d\lambda}{dt} .$$

Now, since $e_{in} = E_m \sin \omega t$, we can write that

$$\frac{d^3\lambda}{dt^3} = \frac{E_m \sin \omega t}{LC} - \frac{R}{L}\frac{d^2\lambda}{dt^2} - \frac{d\lambda}{dt}\left(\frac{L+20\lambda^4 L+1}{LC}\right) - \frac{R(\lambda+4\lambda^5)}{LC}$$

Assume:

$$\lambda \triangleq y_1$$

$$\dot{\lambda} \triangleq \frac{dy_1}{dt} = \dot{y}_1 = y_2$$

$$\ddot{\lambda} \triangleq \frac{d^2y_1}{d^2t} = \ddot{y}_1 = \dot{y}_2 = y_3$$

$$\dddot{\lambda} \triangleq \frac{dy_3{}^3}{dt^3} = \frac{d^3\lambda}{dt^3} = \frac{E_m \sin \omega t}{LC} - \frac{R}{L}\frac{d^2\lambda}{dt^2} - \frac{d\lambda}{dt}\left(\frac{L+20\lambda^4 L+1}{LC}\right) - R\left(\frac{\lambda+4\lambda^5}{LC}\right)$$

Therefore,

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = y_3$$

$$\dot{y}_3 = \frac{E_m \sin \omega t}{LC} - \frac{R}{L}y_3 - y_2\left(\frac{L+20y_1^4 L+1}{LC}\right) - R\left(\frac{y_1+4y_1^5}{LC}\right)$$

or

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{R}{LC} & -\frac{(L+1)}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \frac{20y_1^4 y_2 L + 4Ry_1^5}{LC} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{E_m \sin\omega t}{LC} \end{bmatrix}$$

b. Critical parameters of the pi circuit Before any direct prediction of transformer voltage at ferroresonance can begin, it is necessary to determine first the following relevant factors:

1. Circuit's critical input voltage (E_c) above which ferroresonance occurs.
2. Range of circuit's critical parameters causing ferroresonance.
3. One set of circuits critical parameters R, L, and C.

However, using the Gear package, the items above cannot readily be determined. Therefore, a great number of R, L, C values and circuit input voltages must be tried before it is possible to determine whether or not ferroresonance will occur across the transformer. This means that the method neither provides the physical insight into the phenomenon that is usually acquired by the application of the Describing Function nor does it make available the necessary information without an exhausting process of blind trial-and-error procedure which is both time consuming and unsatisfying.

While the circuit parameters R, L, and C are assumed to be:

$R = 9.02\ \Omega$ (0.002537 P.U.)

$L = 3.15$ mH (0.0335069 P.U.)

$C = 35\ \mu F$ (46.911916) P.U.)

Usually, the critical R, L, C, and E_m values are interrelated and have to be found by this exhaustive trial-and-error procedure. However, since the objective here is to validate theoretical results, we would use the critical values already determined to make a common base for our comparisons.

c. Partial derivatives In addition to the state equation coefficients, one must provide the partial derivatives as illustrated below:

```
PD(1,1) = 0.0 DO
PD(1,2) = 1.0 DO
PD(1,3) = 0.0 DO
PD(2,1) = 0.0 DO
PD(2,2) = 0.0 DO
PD(2,3) = 1.0 DO
PD(3,1) = -R/(L*C)-80.*(1/C)*(y(1)**3)*y(2)
            -20.*(y(1)**4)*R/(L*C)
PD(3,2) = -(L+1.)/(L*C)-20.*(1/C)*y(1)**4
PD(3,3) = -1*(R/L)
```

d. Initial conditions To complete problem formulation, the initial conditions are assumed to be given by:

$y_1(0)$	$y_2(0)$	$y_3(0)$
0.0	1.0	0.0

e. Magnetization curve representation Assuming that G. W. Swift's model, representing the core characteristic of the experimental transformer is given by:

$$I_L = \lambda + 4\lambda^5$$

f. Pi circuit peak input voltage The amplitude of the input to the pi circuit is arbitrarily taken as:

$$E_m = 1,2,3,4,5,6,\ldots\ 30 \text{ and } 31 \text{ V in one volt steps}$$

g. Relative error bound, EPS Used error bound given by: $EPS = 10^{-4}$.

2. Program's output

Using Gear's program, we hope to obtain transformer jump voltage, pi circuit critical input voltage, and ferroresonance switching time. Similar to what was done in the previous experiments, the circuit's input voltage supply E_m in Gear's program was successively incremented by one volt at a time and the corresponding transformer voltage calculated and plotted for 40 cycles. As the input voltage was increased from 1 to 30 V peak, the corresponding steady state transformer voltage appeared sinusoidal and increased gradually and linearly to a peak of about 0.5 P.U. at E_m of 30 V. However, as the input voltage was increased from 30 to 31 V peak, the steady state transformer voltage jumped into a different waveform.

Program's output is:

a. Flux linkage, $\lambda(t)$

b. Voltage, $V_T(t)$

c. Voltage rate of change, $\frac{dV_T(t)}{dt}$

3. Characteristic of the nonlinear mode

As the input voltage was increased from 30 to 31 V peak, the steady state transformer voltage jumped into a distinctively different waveform in the following aspects:

a. Steady state peak jumped from about 0.5 P.U. peak at E_m of 30 V to 1.5 P.U. peak at 31 V.

b. Waveform of transformer voltage contained harmonics.

c. Appearance of a small D.C. offset occurred in transformer voltage.

d. The transformer voltage appeared to have a modulated amplitude at a frequency of about 12 Hz.

4. Low frequency amplitude modulating signal

However, since in our experiments we did not observe the existence of any low frequency ampltidue modulating voltage, it was necessary to trace its origin. In order to do this, the following variations in the Gear input data was attempted:

a. Relative Error Bound Test   To determine whether this modulating signal was part of the solution or an error generated by the program, values of EPS in the Gear program chosen were 10^{-4}, 10^{-6}, and 10^{-10}. In all these three cases, the corresponding solution remained practically the same. This means that the low frequency modulating signal was indeed part of the solution provided by the Gear's program.

EPS	Low frequency modulating signal peak (P.U.)
10^{-4}	0.1315
10^{-6}	0.1315
10^{-10}	0.1315

b. Initial conditions   Same solutions were obtained when different sets of initial conditions were tried:

Initial conditions (P.U.)			Modulating signal peak (P.U.)
$y_1(0)$	$y_2(0)$	$y_3(0)$	
0	1	0	0.1315
0.68	$0.256.10^{-4}$	0	0.1315
0.5129	0.58	-0.4792	0.1315

c. Modified magnetization curve representation

Additional nonlinear cubic term with a variable coefficient (a_3) was added to modify the original representation assumed by G. W. Swift (26). This new representation required the modification of the original third order differential equation as follows:

$$I_L = \lambda + a_3\lambda^3 + 4\lambda^5,$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = y_3$$

$$\dot{y}_3 = \frac{E_m \sin \omega t}{LC} - \frac{RC}{LC}y_3 - R\left(\frac{y_1+a_3y_1^3+4y_1^5}{LC}\right)$$

$$-\left(\frac{L+1+3a_3y_1^2L+20Ly_1^4}{LC}\right)y_2$$

PD(1,1) = 0.0 DO

PD(1,2) = 1.0 DO

PD(1,3) = 0.0 DO

PD(2,1) = 0.0 DO

PD(2,2) = 0.0 DO

PD(2,3) = 1.0 DO

$$PD(3,1) = -R\left(\frac{1+3a_3y_1^2+20y_1^4}{LC}\right) - \left(\frac{6a_3Ly_1y_2+80Ly_1^3y_2}{LC}\right)$$

$$PD(3,2) = -\left(\frac{L+1+3a_3y_1^2L+20Ly_1^4}{LC}\right)$$

$$PD(3,3) = -\frac{R}{L}$$

The effect of $a_3\lambda^3$ term on modulating signal amplitude of the nonlinear mode was measured as seen below:

a_3	Modulating signal amplitude (P.U.)
-3	output returned to linear mode
-2	output returned to linear mode
-1	output returned to linear mode
-0.5	output returned to linear mode
0	0.1315
+1	0.096
+2	0.0915
+4	0.084
+6	0.06425
+9	0.0565

It appears from these data that the modulating signal amplitude can be reduced to a practically negligible value by the addition of the cubic term with an appropriate positive coefficient. The absence of the modulating signal from our experiment results indicates that the magnetization curve representation of our circuit is closer to $\lambda+9\lambda^3+4\lambda^5$ than the term $\lambda+\lambda^5$ used by G. W. Swift.

A comparison of results obtained by both methods using the same circuit parameters is presented here, with

R = 9.02 (0.002537 P.U.),

L = 316 mH (0.0335069 P.U.), and

C = 35 μF (46.911916 P.U.).

Method	Magnetization curve representation
Gear's program	$I_L = \lambda+4\lambda^5$
Present analysis	$N(\lambda_m)$

Method	TRF peak voltage (P.U.) Jump from	Jump to	Circuit's critical input peak voltage (P.U.)	TRF voltage transient time
Gear's program	0.5	1.5[1]	0.27	550 ms
Present analysis	0.8	1.7	0.38	67.3 ms
Experimental	0.96	1.34	0.59	70 ms

Method	Modified magnetization curve representation
Gear's program	$I_L = \lambda+9\lambda^3+4\lambda^5$
Present analysis	$N(\lambda_m)$

[1]Includes amplitude modulating signal peak of 0.054 P.U.

Method	TRF peak voltage (P.U.) Jump from	Jump to	Circuit's critical input peak voltage (P.U.)	TRF voltage transient time
Gear's program	0.33	1.226[1]	0.18	567 ms
Present analysis	0.8	1.7	0.38	67.3 ms
Experimental	0.96	1.34	0.59	70 ms

As seen from the data, the modification of Swift's model removed practically the amplitude modulating signal from transformer voltage and improved on jump-to voltage value. However, the jump-from voltage and the critical supply voltage values became less accurate compared to the experimental value. Also, there is a discrepancy between Gear's predicted voltage transient time and that of the experiment as seen from Figures 63 and 64. It is believed that these discrepancies are due to the approximate magnetization core representation by $\lambda+9\lambda^3+4\lambda^5$. However, we would have obtained better results if we had the magnetization curve represented by a polynomial equation matched accurately to the core characteristic of the transformer instead of $\lambda+9\lambda^3+4\lambda^5$.

---

[1]Includes amplitude modulating signal peak of 0.054 P.U.

IV. METHODOLOGY

A. Configuration of Circuits

To obtain a typical power system inductance value for the pi and series circuits, two well-type inductors (I-701 and I-702) were connected in series to measure a reasonable value for the experiment. The inductance of 316 mH was measured when the core of the inductor was on the bottom of the well. Resistance of this inductor (9.02 Ω) was calculated from measurement of the current through the inductor (0.9 Amp RMS) and inductor power (7.31 watt):

$$P = I^2 R_x$$

$$R_x = \frac{7.31}{0.81}$$

$$R_x = 9.02\ \Omega$$

Three oil filled capacitor banks were used, 250 μF each, with toggle switches for 10 and 50 μF increments. The investigated line transformer is rated 3 KVA, 2400/240V. These values of R = 9.02 Ω and L = 316 mH are used in the pi and series circuits of Figures 1 and 2, respectively.

B. Experimental Determination of Magnetization Curve

The magnetization curve was experimentally determined for the 3 KVA line transformer using the following operational amplifier circuit:

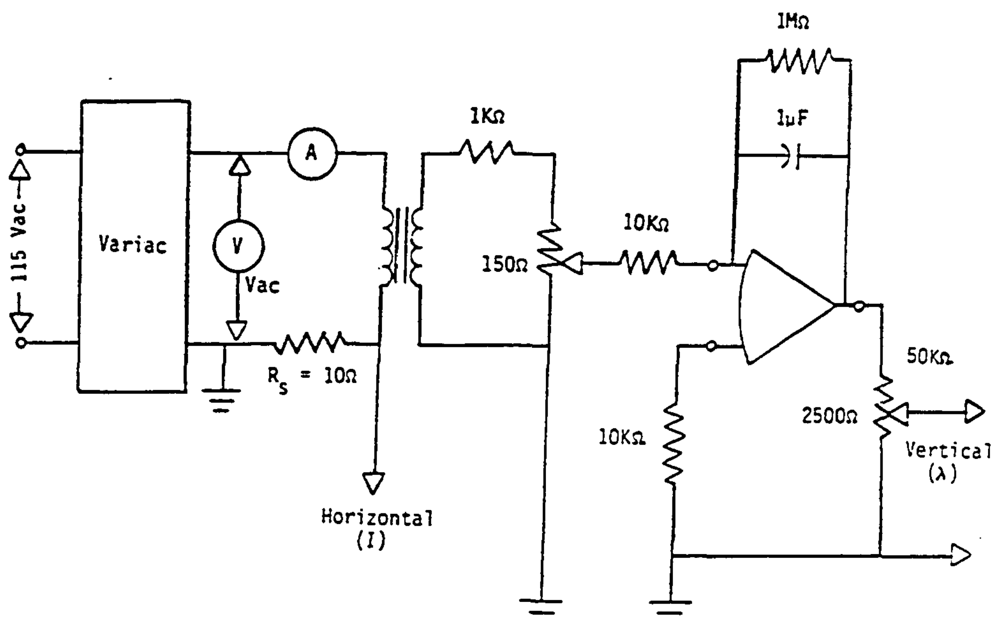


Figure 12. Circuit for determining transformer's magnetization curve

$$I_H(\text{Amp}) = \frac{V_H}{R_S} = \frac{V_H}{10}$$

$$(\text{WT/Amp}) = \frac{100}{20}\int v_c dt = 5\int v_c dt$$

Vac(V-RMS)	Reading		Scale		Calculated	
	V	H	V	H	λ	I
66	3.6D	4.1D	0.2/D	1/D	3.6	0.41
75	1.7D	5D	0.5/D	1/D	4.25	0.5
85	1.82D	3.4D	0.5/D	2/D	4.55	0.68
95	2.1D	5.1D	0.5/D	2/D	5.25	1.02
105	2.3D	3D	0.5/D	5/D	5.75	1.5
115	2.5D	4.5D	0.5/D	5/D	6.25	2.25
125	2.7D	3.4D	0.5/D	10/D	6.75	3.4
135	2.95D	5.2D	0.5/D	10/D	7.375	5.2

The magnetization characteristic of the 3 kVA line transformer was then plotted as shown in Figure 13.

It can be assumed that the magnetic characteristic of a power transformer has three regions:

1. Unsaturated (initial) region: The first linear part that is in close agreement with the actual curve near the origin.

2. Saturated region: The second linear part that is in close agreement with the actual curve near the saturation.

3. Transition region: It is the transitional region between the two linear parts. This is the region where the most error occurs between the actual and the two linear parts.

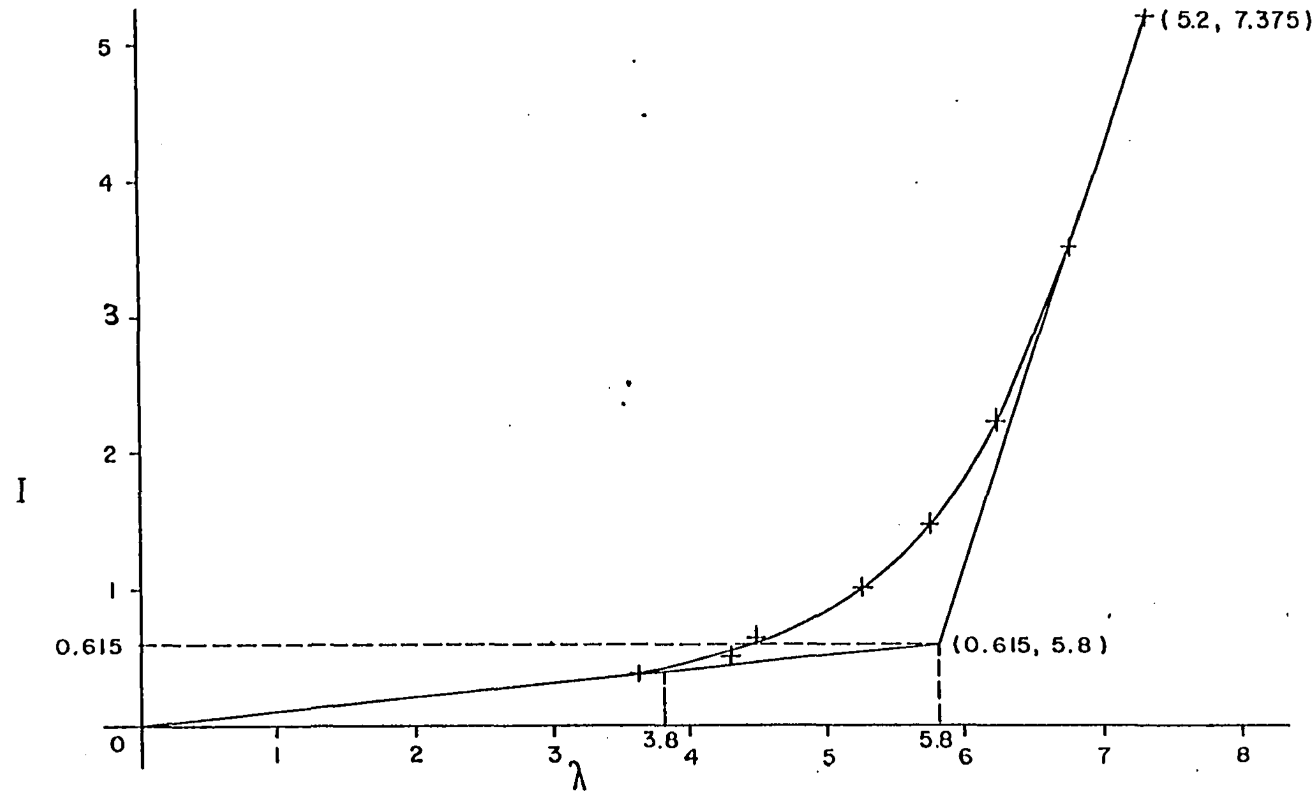


Figure 13. Experimentally determined magnetization curve

C. Mathematical Model of Power Transformer

1. Pi circuit transformer Describing Function

The asymptotic lines were drawn, one for the unsaturated (initial) and the other for the saturated region. These asymptotes intersected at the point $\lambda = 5.8$, $I = 0.615$.

$$m_1 = \frac{0.615}{5.8} = 0.1060345$$

$$m_1 = m_B$$

$$m_2 = \frac{4.585}{1.575} = 2.9111111$$

$$\therefore m_1 = 1 \text{ P.U.}$$

$$m_2 = 27.454377 \text{ P.U.}$$

$$i_T = m_2\lambda + I_0$$

$$i_T = 0, \lambda = \lambda_1$$

$$\therefore 0 = m_2\lambda_1 + I_0$$

$$I_0 = -m_2\lambda_1 \quad \text{or} \quad \lambda_1 = -\frac{I_0}{m_2}$$

$$m_2 = \frac{m_1\lambda_0}{\lambda_0 - \lambda_1}$$

$$\therefore m_2 = \frac{m_1\lambda_0}{\lambda_0 + \frac{I_0}{m_2}}$$

$$m_1\lambda_0 = m_2\lambda_0 + I_0$$

or

$$I_0 = (m_1 - m_2)\lambda_0$$

Substituting in the above equation:

$$\therefore i_T = m_2\lambda(t) + (m_1 - m_2)\lambda_0$$

$$\therefore i_T = m_1\lambda(t) \qquad 0<\theta\leq\theta_0 \qquad (4.1)$$

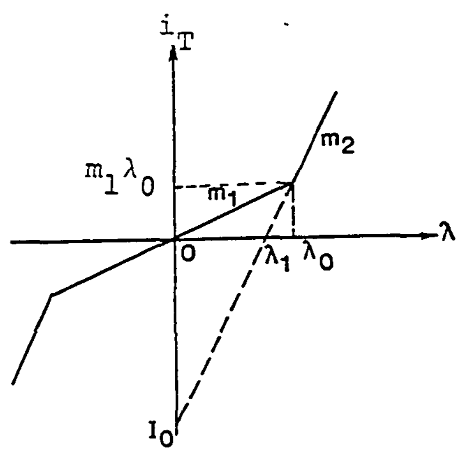


Figure 14. Two-slope representation of magnetization curve in a pi circuit

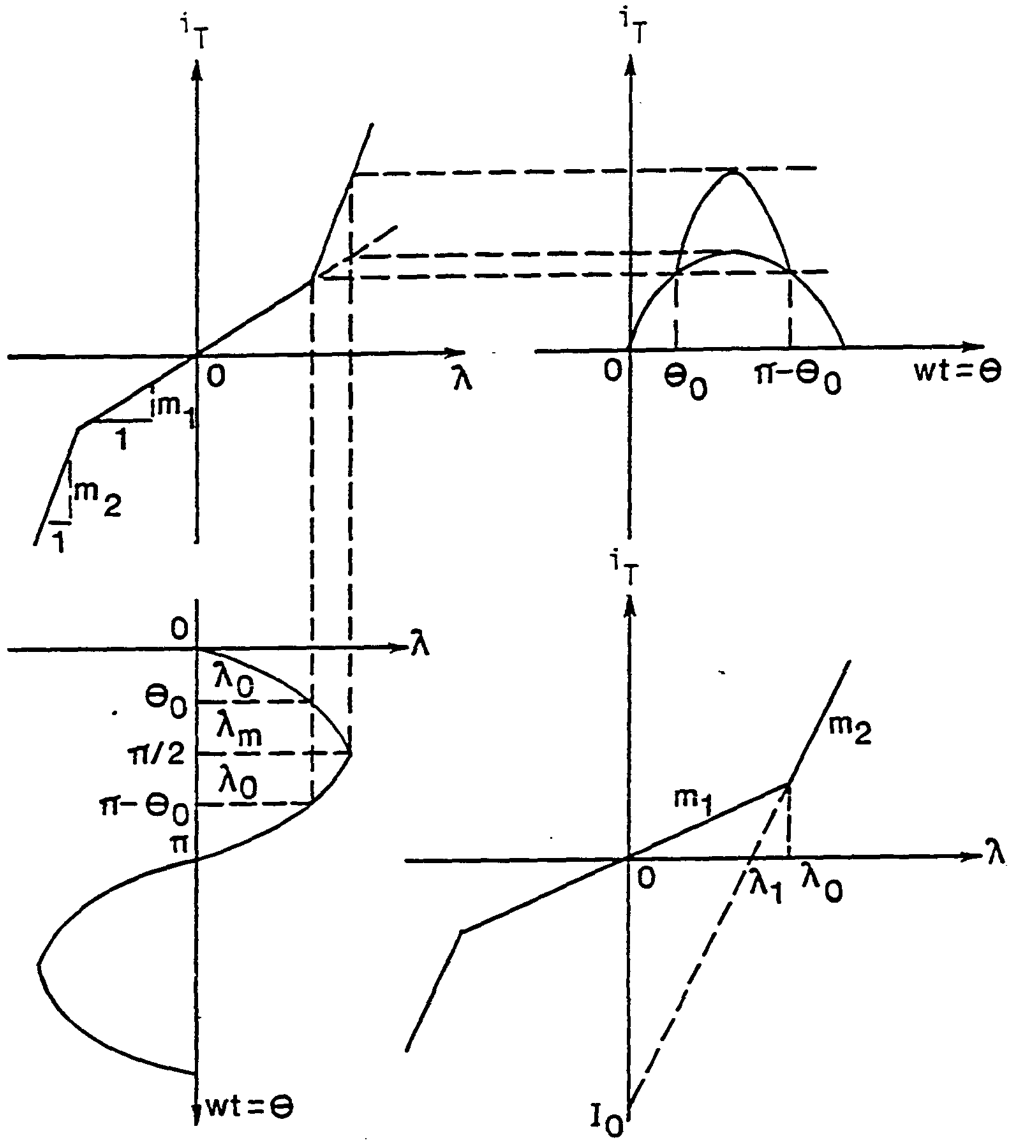


Figure 15. Nonlinear mode transformer current in a pi circuit

and

$$i_L = (m_1-m_2)\lambda_0 + m_2\lambda(t) \qquad \theta_0<\theta\leq\pi/2 \tag{4.2}$$

where knee of the curve is λ_0 and

$$\lambda_0 = \frac{3.8}{5.8}$$

$$= 0.6551724 \text{ P.U.}$$

The nonlinearity is assumed to be static, single valued, odd, and sinusoidal input Describing Function. Therefore, by definition of the Describing Function (DF), $N(\lambda_m)$:

$$N(\lambda_m) = \frac{4}{\pi\lambda_m} \int_0^{\frac{\pi}{2}} f\,(\lambda_m \sin\,\theta)\sin\,\theta\;d\theta$$

$$= \frac{4}{\pi\lambda_m}\left[\int_0^{\theta_0} m_1\lambda(t)\sin\theta\;d\theta + \int_{\theta_0}^{\frac{\pi}{2}}\{(m_1-m_2)\lambda_0 + \right.$$

$$\left. + m_2\lambda(t)\}\;\sin\;\theta\;d\theta\right]$$

Since $\lambda(t) = \lambda_m \sin\,\theta$

$$N(\lambda_m) = \frac{4}{\pi\lambda_m}\left[\int_0^{\theta_0} m_1(\lambda_m\sin\theta)\sin\theta\;d\theta + \int_{\theta_0}^{\frac{\pi}{2}}\{(m_1-m_2)\lambda_0\right.$$

$$\left. + m_2(\lambda_m\sin\,\theta)\}\sin\theta\;d\theta\right]$$

$$= \frac{4}{\pi\lambda_m}\left[\int_0^{\theta_0} m_1\lambda_m \sin^2\theta\, d\theta + \int_{\theta_0}^{\frac{\pi}{2}} \{(m_1-m_2)\lambda_0 \sin\theta\right.$$

$$\left. + m_2\lambda_m \sin^2\theta\, d\theta\} d\theta\right]$$

$$= \frac{4}{\pi\lambda_m}\left[\int_0^{\theta_0} m_1\lambda_m\left(\frac{1-\cos 2\theta}{2}\right) d\theta\right.$$

$$\left. + \int_{\theta_0}^{\frac{\pi}{2}} \{(m_1-m_2)\lambda_0 \sin\theta\, d\theta + m_2\lambda_m\left(\frac{1-\cos 2\theta}{2}\right) d\theta\right]$$

$$= \frac{2m_1}{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\theta_0} - \frac{4}{\pi\lambda_m}\left[(m_1-m_2)\lambda_0 \cos\theta\right]_{\theta_0}^{\frac{\pi}{2}} + \frac{2m_2}{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta_0}^{\frac{\pi}{2}}$$

$$= \frac{2m_1}{\pi}\left[\theta_0 - \frac{\sin 2\theta_0}{2}\right] + \frac{4}{\pi\lambda_m}(m_1-m_2)\lambda_0\left[-\cos\frac{\pi}{2} + \cos\theta_0\right]$$

$$+ \frac{2m_2}{\pi}\left[\frac{\pi}{2} - \frac{\sin 2(\frac{\pi}{2})}{2}\right] - \frac{2m_2}{\pi}\left[\theta_0 - \frac{\sin 2\theta_0}{2}\right]$$

$$N(\lambda_m) = \frac{2m_1\theta_0}{\pi} - \frac{2m_2\theta_0}{\pi} + m_2 - \frac{2m_1}{\pi}\,\frac{\sin 2\theta_0}{2} + \frac{2m_2}{\pi}\,\frac{\sin 2\theta_0}{2}$$

$$+ \frac{4}{\pi}(m_1-m_2)\frac{\lambda_0}{\lambda_m}\cos\theta_0$$

$$N(\lambda_m) = \frac{2(m_1-m_2)}{\pi}\theta_0 - \frac{2(m_1-m_2)}{\pi}\left[\frac{\sin 2\theta_0}{2} - 2\frac{\lambda_0}{\lambda_m}\cos\theta_0\right] + m_2$$

$$= \frac{2(m_1-m_2)}{\pi}\left[\theta_0 - \frac{\sin 2\theta_0}{2} + 2\frac{\lambda_0}{\lambda_m}\cos\theta_0\right] + m_2$$

since

$$\lambda_0 = \lambda_m \sin\theta_0$$

$$\frac{\lambda_0}{\lambda_m} = \sin\,\theta_0$$

$$\theta_0 = \sin^{-1}\frac{\lambda_0}{\lambda_m}$$

and

$$\frac{\sin 2\theta_0}{2} = \frac{2\sin\theta_0\,\cos\theta_0}{2}$$

$$= \sin\theta_0\,\cos\theta_0$$

$$\therefore N(\lambda_m) = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} - \sin\theta_0\,\cos\theta_0 + \frac{2\lambda_0}{\lambda_m}\cos\theta_0] + m_2$$

$$= \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} - \cos\theta_0(\frac{\lambda_0}{\lambda_m} - \frac{2\lambda_0}{\lambda m})] + m_2$$

$$= \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} - \cos\theta_0(-\frac{\lambda_0}{\lambda_m})] + m_2$$

$$= \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} + \frac{\lambda_0}{\lambda_m}\cos\theta_0] + m_2$$

$$= \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} + \frac{\lambda_0}{\lambda_m}\sqrt{1-(\frac{\lambda_0}{\lambda_m})^2}] + m_2$$

$$N(\lambda_m) = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} + \{(\frac{\lambda_0}{\lambda_m})^2 - (\frac{\lambda_0}{\lambda_m})^4\}^{\frac{1}{2}}] + m_2$$

$$\frac{d\sin^{-1}}{d\lambda_m}\left(\frac{\lambda_0}{\lambda_m}\right) = \frac{1}{\sqrt{1 - \left(\frac{\lambda_0}{\lambda_m}\right)^2}} \frac{d}{d\lambda_m}\left(\frac{\lambda_0}{\lambda_m}\right)$$

$$\frac{dN(\lambda_m)}{d\lambda_m} = \frac{2(m_1-m_2)}{\pi}\left[\frac{-\frac{\lambda_0}{\lambda_m^2}}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_m}\right)^2}} + \frac{1}{2}\left\{\left(\frac{\lambda_0}{\lambda_m}\right)^2\right.\right.$$

$$\left.\left. - \left(\frac{\lambda_0}{\lambda_m}\right)^4\right\}^{-\frac{1}{2}}\left(-2\frac{\lambda_0^2}{\lambda_m^3} + 4\frac{\lambda_0^4}{\lambda_m^5}\right)\right]$$

$$\dot{N}(\lambda_m) = \frac{2(m_1-m_2)}{\pi}\left[\frac{-\left(\frac{\lambda_0}{\lambda_m}\right)^2 \frac{1}{\lambda_0}}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_m}\right)^2}} - \frac{\left(\frac{\lambda_0}{\lambda_m}\right)^2 \frac{1}{\lambda_0}}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_m}\right)^2}}\right.$$

$$\left. + \frac{2\left(\frac{\lambda_0}{\lambda_m}\right)^4 \frac{1}{\lambda_0}}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_m}\right)^2}}\right]$$

$$= \frac{2(m_1-m_2)}{\pi}\left[\frac{-2\left(\frac{\lambda_0}{\lambda_m}\right)^2\frac{1}{\lambda_0}}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_m}\right)^2}} + \frac{2\left(\frac{\lambda_0}{\lambda_m}\right)^4\frac{1}{\lambda_0}}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_m}\right)^2}}\right]$$

$$= \frac{-4(m_1-m_2)}{\pi} \frac{(\frac{\lambda_0}{\lambda_m})^2 \frac{1}{\lambda_0}}{\sqrt{1-(\frac{\lambda_0}{\lambda_m})^2}} [1-(\frac{\lambda_0}{\lambda_m})^2]$$

$$\dot{N}(\lambda_m) = \frac{-4(m_1-m_2)}{\pi} (\frac{\lambda_0}{\lambda_m})^2 \frac{1}{\lambda_0} \sqrt{1-(\frac{\lambda_0}{\lambda_m})^2}$$

$$\lambda_m[\dot{N}(\lambda_m)] = \frac{-4(m_1-m_2)}{\pi} (\frac{\lambda_0}{\lambda_m}) \sqrt{1-(\frac{\lambda_0}{\lambda_m})^2}$$

In using the Describing Function Technique, the transitional region error is minimized and the small signal constraint of the piecewise linearization is lifted but the advantage of linear approximation is retained. What must be done now is to determine the operation performed by the nonlinear element (Power Transformer) on the input signal of finite size, and approximate this in some way by a linear operation.

Then, the magnetization characteristic of the power transformer may be represented by the odd, single-valued, static, and sinusoidal Describing Function, $N(\lambda_m)$:

$$N(\lambda_m) = \frac{2(m_1-m_2)}{\pi} [\sin^{-1} \frac{\lambda_0}{\lambda_m} + \frac{\lambda_0}{\lambda_m} \sqrt{1-(\frac{\lambda_0}{\lambda_m})^2}] + m_2 \qquad (4.3)$$

where

$\lambda_m \geq \lambda_0$

Assume the input to the nonlinearity $N(\lambda_m)$ is λ_m, the magnitude of the fundamental amplitude of the output is:

$$\text{Output} = \text{Input} \times N(\lambda_m)$$

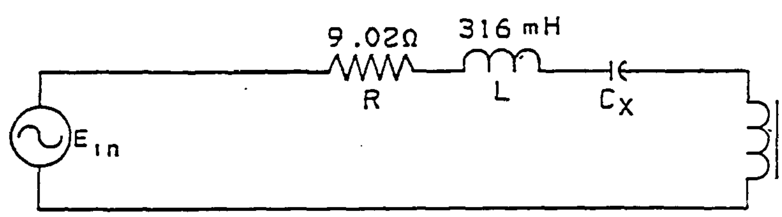
since

$$L_{eg} = \frac{\text{Flux linkage}}{\text{Current}}$$

$$L_{eg} = \frac{\lambda_m}{\lambda_m N(\lambda_m)} \tag{4.4}$$

$$L_{eg} = \frac{1}{N(\lambda_m)}$$

<u>2. Series circuit Transformer Describing Function $N(I_m)$</u>



In the series circuit configuration of Figure 2, the same calculated values for resistance (9.02 Ω) and measured inductance (316 mH) were used. For mathematical convenience, the magnetization curve is rotated, where

$$m_2 = \frac{1}{2.9111111} = 0.3435115$$

$$m_1 = \frac{1}{0.1060345} = 9.4308943$$

$$m_1 = m_B$$

$$\therefore\ m_1 = 1 \text{ P.U.}$$

$$m_2 = 0.0364241 \text{ P.U.}$$

$$\lambda(t) = m_2 i_T(t) + \lambda_1 \qquad (4.5)$$

Since $\lambda(t) = \lambda_0$ at $i_T = I_0$

$$\frac{\lambda_0 - \lambda_1}{I_0} = m_2 \qquad (4.6)$$

$$\frac{\lambda_0 - \lambda_1}{I_0 - I_1} = m_1 \qquad (4.7)$$

$$\frac{\lambda_1}{I_1} = m_1 \qquad (4.8)$$

Solving Equations 4.6 and 4.7, we get:

$$\lambda_0 - \lambda_1 = I_0 m_2$$

$$\lambda_0 - \lambda_1 = m_1(I_0 - I_1)$$

$$\therefore\ I_0 m_2 = m_1(I_0 - I_1)$$

$$I_0 m_2 = m_1 I_0 - m_1 I_1$$

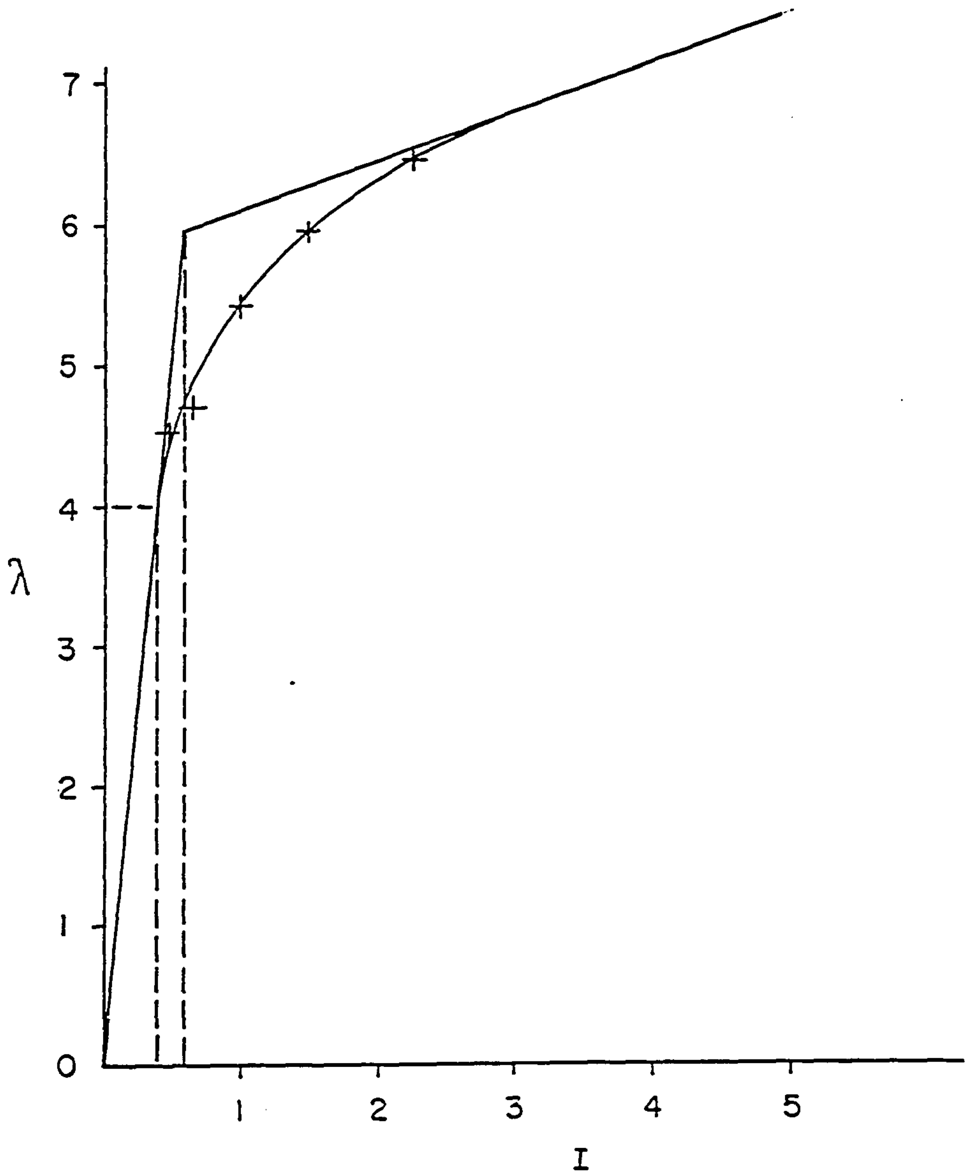


Figure 16. Experimentally determined magnetization curve rotated for mathematical convenience

$$m_1I_1 = m_1I_0 - I_0m_2$$

$$m_1I_1 = I_0(m_1-m_2)$$

$$I_1 = \frac{I_0(m_1-m_2)}{m_1}$$

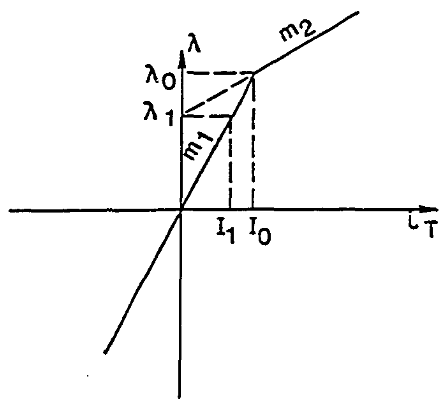


Figure 17. Two-slope representation of magnetization curve in a series circuit

Using Equation 4.8, we obtain:

$$\lambda_1 = I_1m_1$$

$$\lambda_1 = \frac{I_0(m_1-m_2)}{m_1} \cdot m_1$$

$$\lambda_1 = I_0(m_1-m_2)$$

Substituting in Equation 4.5:

$$\therefore \; \lambda(t) = m_2 i_T(t) + I_0(m_1 - m_2)$$

$$\therefore \; \lambda(t) = m_1 i(t) \qquad 0 \le \theta \le \theta_0 \tag{4.9}$$

$$\lambda(t) = (m_1 - m_2) I_0 + m_2 i(t) \qquad \theta_0 \le \theta \le \frac{\pi}{2} \tag{4.10}$$

where knee of the curve is I_0 and

$$I_0 = \frac{0.4}{0.615} = 0.6504065 \text{ P.U.}$$

By definition of the static, single valued, odd, and sinusoidal input Describing Function, $N(I_m)$:

$$N(I_m) = \frac{4}{\pi I_m} \int_0^{\frac{\pi}{2}} f(I_m \sin\theta) \sin\theta \, d\theta$$

$$N(I_m) = \frac{4}{\pi I_m} \Big[\int_0^{\theta_0} m_1 i(t) \sin\theta \, d\theta + \int_{\theta_0}^{\frac{\pi}{2}} \{(m_1 - m_2) I_0 + m_2 i(t)\} \sin\theta \, d\theta \Big]$$

Assume $i(t) = I_m \sin\theta$

$$N(I_m) = \frac{4}{\pi I_m} \Big[\int_0^{\theta_0} m_1 (I_m \sin\theta) \sin\theta \, d\theta + \int_{\theta_0}^{\frac{\pi}{2}} \{(m_1 - m_2) I_0 + m_2 (I_m \sin\theta)\} \sin\theta \, d\theta \Big]$$

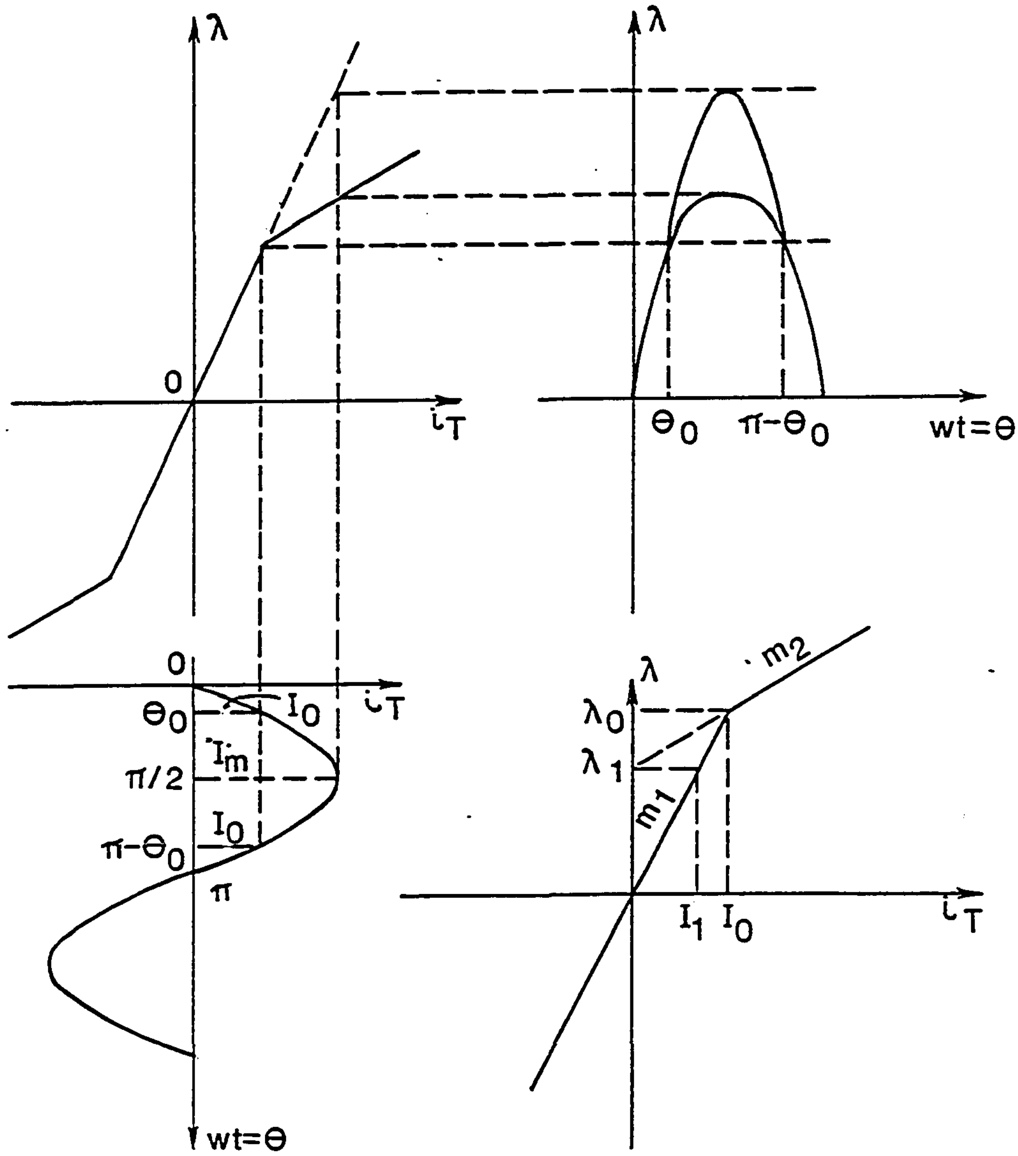


Figure 18. Transformer voltage of the nonlinear mode in a series circuit

$$N(I_m) = \frac{4}{\pi I_m}\left[\int_0^{\theta_0} m_1 I_m \sin^2\theta \, d\theta + \int_{\theta_0}^{\frac{\pi}{2}} (m_1 - m_2) I_0 \sin\theta \, d\theta + \int_{\theta_0}^{\frac{\pi}{2}} m_2 I_m \sin^2\theta \, d\theta\right]$$

$$N(I_m) = \frac{4}{\pi I_m}\left[\int_0^{\theta_0} m_1 I_m \left(\frac{1-\cos 2\theta}{2}\right) d\theta + \int_{\theta_0}^{\frac{\pi}{2}} (m_1 - m_2) I_0 \sin\theta \, d\theta + \int_{\theta_0}^{\frac{\pi}{2}} m_2 I_m \left(\frac{1-\cos 2\theta}{2}\right) d\theta\right]$$

$$N(I_m) = \frac{2m_1}{\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\theta_0} \frac{4}{\pi I_m}\left[(m_1 - m_2) I_0 \cos\theta\right]_{\theta_0}^{\frac{\pi}{2}} + \frac{2m_2}{\pi}\left[\theta - \sin 2\theta\right]_{\theta_0}^{\frac{\pi}{2}}$$

$$N(I_m) = \frac{2m_1}{\pi}\left[\theta_0 - \frac{\sin 2\theta_0}{2}\right] + \frac{4}{\pi I_m}(m_1 - m_2) I_0 \left[-\cos\frac{\pi}{2} + \cos\theta_0\right] + \frac{2m_2}{\pi}\left[\frac{\pi}{2} - \frac{\sin 2(\frac{\pi}{2})}{2}\right] - \frac{2m_2}{\pi}\left[\theta_0 - \frac{\sin 2\theta_0}{2}\right]$$

$$N(I_m) = \frac{2m_1\theta_0}{\pi} - \frac{2m_2\theta_0}{\pi} + m_2 - \frac{2m_1}{\pi}\frac{\sin 2\theta_0}{2} + \frac{2m_2}{\pi}\frac{\sin 2\theta_0}{2} + \frac{4}{\pi}(m_1 - m_2)\frac{I_0}{I_m}\cos\theta_0$$

$$N(I_m) = \frac{2(m_1-m_2)}{\pi}\theta_0 - \frac{2(m_1-m_2)}{\pi}[\frac{\sin 2\theta_0}{2} + 2\frac{I_0}{I_m}\cos\theta_0] + m_2$$

$$N(I_m) = \frac{2(m_1-m_2)}{\pi}[\theta_0 - \frac{\sin 2\theta_0}{2} + 2\frac{I_0}{I_m}\cos\theta_0] + m_2$$

$$i = I_m \sin\theta$$

$$I_0 = I_m \sin\theta_0$$

$$\frac{I_0}{I_m} = \sin\theta_0$$

$$\theta_0 = \sin^{-1}\frac{I_o}{I_m}$$

and

$$\frac{\sin 2\theta_0}{2} = \frac{2\sin\theta_0 \cos\theta_0}{2}$$

$$= \sin\theta_0 \cos\theta_0$$

$$\therefore N(I_m) = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{I_0}{I_m} - \sin\theta_0\cos\theta_0 + \frac{2I_0}{I_m}\cos\theta_0]$$

$$+ m_2$$

$$= \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{I_0}{I_m} - \cos\theta_0(\frac{I_0}{I_m} - \frac{2I_0}{I_m})] + m_2$$

$$= \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{I_0}{I_m} + \frac{I_0}{I_m}\cos\theta_0] + m_2$$

$$N(I_m) = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{I_0}{I_m} + \frac{I_0}{I_m}\sqrt{1 - (\frac{I_0}{I_m})^2}] + m_2$$

$$N(I_m) = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{I_0}{I_m} + \{(\frac{I_0}{I_m})^2 - (\frac{I_0}{I_m})^4\}^{\frac{1}{2}} + m_2$$

$$\frac{d\sin^{-1}\left(\frac{I_0}{I_m}\right)}{dI_m} = \frac{1}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}}\ \frac{d}{dI_m}\left(\frac{I_0}{I_m}\right)$$

$$\frac{dN(I_m)}{dI_m} = \frac{2(m_1-m_2)}{\pi}\left[\frac{-\frac{I_0}{I_m^2}}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}}\right.$$

$$\left. + \frac{1}{2}\left\{\left(\frac{I_0}{I_m}\right)^2 - \left(\frac{I_0}{I_m}\right)^4\right\}^{-\frac{1}{2}}\left(-2\frac{I_0^2}{I_m^3} + 4\frac{I_0^4}{I_m^5}\right)\right]$$

$$\dot{N}(I_m) = \frac{2(m_1-m_2)}{\pi}\left[\frac{-\left(\frac{I_0}{I_m}\right)^2\frac{1}{I_0}}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}} - \frac{\left(\frac{I_0}{I_m}\right)^2\frac{1}{I_0}}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}}\right.$$

$$\left. + \frac{2\left(\frac{I_0}{I_m}\right)^4\frac{1}{I_0}}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}}\right]$$

$$= \frac{2(m_1-m_2)}{\pi}\left[\frac{-2\left(\frac{I_0}{I_m}\right)^2\frac{1}{I_0}}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}} + 2\,\frac{\left(\frac{I_0}{I_m}\right)^4\frac{1}{I_0}}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}}\right]$$

$$= \frac{-4(m_1-m_2)}{\pi}\ \frac{\left(\frac{I_0}{I_m}\right)^2\frac{1}{I_0}}{\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}}\left[1-\left(\frac{I_0}{I_m}\right)^2\right]$$

$$= \frac{-4(m_1-m_2)}{\pi}\left(\frac{I_0}{I_m}\right)^2 \frac{1}{I_0}\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}$$

$$I_m\dot{N}(I_m) = \frac{-4(m_1-m_2)}{\pi}\left(\frac{I_0}{I_m}\right)\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}$$

Then, the magnetization characteristic of the power transformer can be represented by the odd, single-valued, static, and sinusoidal input Describing Function, $N(I_m)$:

$$N(I_m) = \frac{2(m_1-m_2)}{\pi}\left[\sin^{-1}\frac{I_0}{I_m} + \frac{I_0}{I_m}\sqrt{1-\left(\frac{I_0}{I_m}\right)^2}\right] + m_2 \qquad (4.11)$$

Assume the input to the nonlinearity $N(I_m)$ is I_m, the magnitude of the fundamental amplitude of the output is:

Output = Input · $N(I_m)$

since

$$L_{eg} = \frac{\text{Flux linkage}}{\text{Current}}$$

$$\therefore L_{eg} = \frac{I_m N(I_m)}{I_m}$$

$$L_{eg} = N(I_m) \qquad (4.12)$$

D. Linear Part Representation of the System

1. Pi circuit $G_p(S,R,L,C)$

Four equations are required to describe completely the shunt-circuit configuration of Figure 1:

$$e_{in}(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{d\lambda(t)}{dt} \tag{4.13}$$

$$\frac{1}{c}\int i_c(t)dt = \frac{d\lambda(t)}{dt} \tag{4.14}$$

$$i(t) = i_T(t) + i_c(t) \tag{4.15}$$

$$i_T(t) = f(\lambda) = \lambda + a_3\lambda^3 + a_5\lambda^5 + \ldots + a_n\lambda^n \tag{4.16}$$

If we assume that equations 4.13, 4.14, and 4.15 are Laplace transformable, then:

$$E_{in}(s) = RI(s) + LSI(s) + S\lambda(s)$$

$$\frac{1}{c}\,\frac{I_c(s)}{s} = s\lambda(s)$$

$$I(s) = I_T(s) + I_c(s)$$

and from Equation 4.13:

$$\lambda(s) = \frac{E_{in}(s)}{s} - \left(\frac{R+LS}{s}\right)I(s)$$

Substituting 4.15 in the above equation:

$$\lambda(s) = \frac{E_{in}(s)}{s} - (\frac{R+LS}{S})[I_T(s)+I_c(s)]$$

$$\lambda(s) = \frac{E_{in}(s)}{s} - (\frac{R+LS}{s})I_T(s) - (\frac{R+LS}{s})I_c(s)$$

Substituting 4.14 in the above equation:

$$\lambda(s) = \frac{E_{in}(s)}{s} - \frac{R+SL}{s}I_T(s) - (\frac{R+LS}{s})[S^2C\lambda(s)]$$

$$\lambda(s) + (R+LS)CS\lambda(s) = \frac{E_{in}(s)}{s} - \frac{R+SL}{s}I_T(s)$$

$$\lambda(s) + RCS\,\lambda(s) + LCS^2\lambda(s) = \frac{E_{in}(s)}{s} - \frac{R+SL}{s}I_T(s)$$

$$\lambda(s)[1 + RCS + LCS^2] = \frac{E_{in}(s)}{s} - \frac{R+SL}{s}I_T(s)$$

$$\therefore \lambda(s) = \frac{E_{in}(s)}{s(1+RCS+LCS^2)} - \frac{R+SL}{S(1+RCS+LCS^2)}I_T(s)$$

Let:

$$G_{in}(s) = \frac{1}{S(1+RCS+LCS^2)} \qquad (4.17)$$

and

$$G_p(s) = \frac{R+LS}{S(1+RCS+LCS^2)} \qquad (4.18)$$

Then, $G_p(s)$ is the linear part of the system representation.

$$\therefore \lambda(s) = G_{in}(s)E_{in}(s) - G_p(s)I_T(s) \qquad (4.19)$$

This equation is represented by the following block diagram:

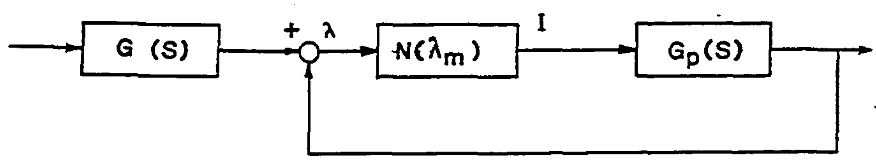


Figure 19. Standard block diagram representation for the pi circuit

If we assume that the linear system $G_p(s)$ satisfies the Filter Hypothesis, i.e., it acts to filter out all significant harmonics generated by the nonlinearity, then the feedback is linear because it is the Fundamental Frequency of the output current of the nonlinearity.

2. Series circuit $G_s(S,R,L,C)$

Two equations are required to describe the series circuit configuration:

$$E_{in} = Ri_R + L\frac{di_T}{dt} + \frac{1}{c}\int i_c dt + \frac{d\lambda(t)}{dt} \quad (4.20)$$

$$i = i_R = i_C = i_L = i_T \quad (4.21)$$

If we assume that the above equations are Laplace transformable, then:

$$E_{in}(S) = RI_R(S) + LSI_L(S) + \frac{1}{CS}I_C(S) + S\lambda(S)$$

$$S\lambda(S) = E_{in}(S) - RI_R(S) - LSI_L(S) - \frac{1}{CS}I_C(S)$$

Substituting 4.21 in the above equation, we obtain:

$$S\lambda(S) = E_{in}(S) - RI_T(S) - LSI_T(S) - \frac{1}{CS}I_T(S)$$

$$S\lambda(S) = E_{in}(S) - [R + LS + \frac{1}{CS}]I_T(S)$$

$$[R + LS + \frac{1}{CS}]I_T(S) = E_{in}(S) - S\lambda(S)$$

$$I_T(S) = \frac{CSE_{in}(S)}{RCS + LCS^2 + 1} - \frac{CS^2\lambda(S)}{RCS + LCS^2 + 1}$$

Let

$$G_{in}(S) = \frac{CS}{LCS^2 + RCS + 1} \qquad (4.22)$$

and

$$G_s(S) = \frac{CS^2}{LCS^2 + RCS + 1} \qquad (4.23)$$

Then, $G_s(S)$ is the linear part of the system representation.

$$\therefore I_T(S) = G_{in}(S)E_{in}(S) - G_s(S)\lambda(S) \qquad (4.24)$$

This equation is represented by the following block diagram:

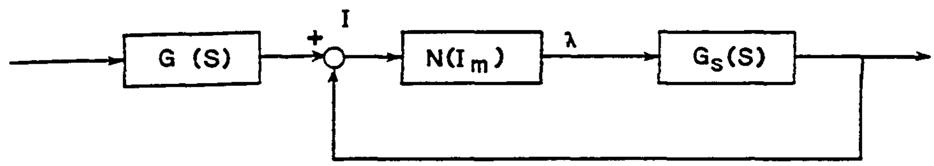


Figure 20. Standard block diagram representation for the series circuit

where

I_T = fundamental frequency input current in the series circuit

$E_{in}(S)$ = input voltage

$\lambda(S)$ = output flux linkage of the nonlinearity

If we assume that the linear system G(S) satisfies the Filter Hypothesis, i.e., it acts to filter out all significant harmonics generated by the nonlinearity, then the feedback is linear because it is the fundamental frequency of the output flux of the nonlinearity.

E. Two-slope Incremental Input Describing Function

1. Pi circuit $N(\lambda_m,\phi)$

For the nonlinear system, the possibility of a jump resonance is studied in terms of a sinusoidal perturbation of frequency about a steady state forced oscillation at the same frequency, i.e., the input to the nonlinearity is made up of two voltage signals, one the steady state input and another with much smaller amplitude but shifted in phase:

$$\lambda(t) = \lambda_m \sin \omega t + \mu \sin(\omega t+\phi) \tag{4.25}$$

where the second term represents the incremental input such that

$$\mu << \lambda m$$

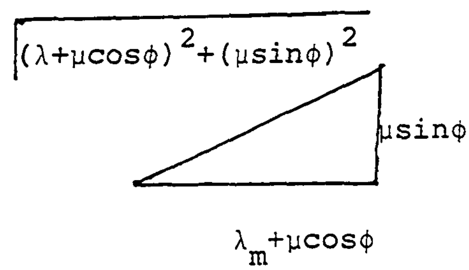
and

$$\begin{aligned}\lambda(t) &= \lambda_m \sin \omega t + \mu \sin \omega t \cos\phi + \mu \cos \omega t \sin\phi \\ &= (\lambda_m + \cos\phi)\sin \omega t + (\mu \sin\phi)\cos \omega t \\ &= \sqrt{(\lambda_m + \cos\phi)^2 + (\mu\sin\phi)^2} \\ &\quad \times \frac{(\lambda_m + \cos\phi)\sin \omega t}{\sqrt{(\lambda_m + \mu\cos\phi)^2 + (\mu\sin\phi)^2}} + \frac{\mu\sin\phi \cos \omega t}{\sqrt{(\lambda_m + \mu\cos\phi)^2 (\mu\sin\phi)^2}}\end{aligned}$$

$$= \sqrt{(\lambda_m+\mu\cos\phi)^2+(\mu\sin\phi)^2}\,[\sin\omega t\cos\gamma + \cos\omega t\sin\gamma]$$

where

$$\gamma = \tan^{-1}\frac{\mu\sin\phi}{\lambda_m+\mu\cos\phi}$$



since

$$\mu << \lambda_m$$

$$\therefore \sqrt{(\lambda_m+\mu\cos\phi)^2 + (\mu\sin\phi)^2} \approx \lambda_m + \mu\cos\phi$$

and

$$\tan^{-1}\frac{\mu\sin\phi}{\lambda_m+\mu\cos\phi} \approx \frac{\mu}{\lambda_m}\sin\phi$$

Hence, the input to nonlinearity can be written as:

$$\lambda(t) = (\lambda_m+\mu\cos\phi)\sin(\omega t + \frac{\mu\sin\phi}{\lambda_m}) \qquad (4.26)$$

If we represent the Describing Function as a complex gain of the transformer input in the pi circuit block diagram of Figure 19, substitute 4.26 in Equation 12.2 and manipulate, we will obtain the Incremental Input Describing Function of Appendix D equation 12.3:

$$N(\lambda_m,\phi) = N(\lambda_m) + \frac{\lambda_m\dot{N}(\lambda_m)}{2}(1+e^{-2j\phi})$$

2. Series circuit $N(I_m,\phi)$

We will assume that the input to the nonlinearity is made up of two current signals, the steady-state input, and another with much smaller amplitude but shifted in phase.

Considering that steady-state input and its increment of the same frequency, we have:

$$i(t) = I_m \sin \omega t + \mu \sin(\omega t+\phi)$$

where the second term represents the incremental input, such that

$$i(t) = I_m \sin wt + \mu \sin wt \cos\phi + \mu \cos wt \sin\phi$$

$$= (I_m+\mu\cos\phi)\sin \omega t + (\mu\sin\phi)\cos \omega t$$

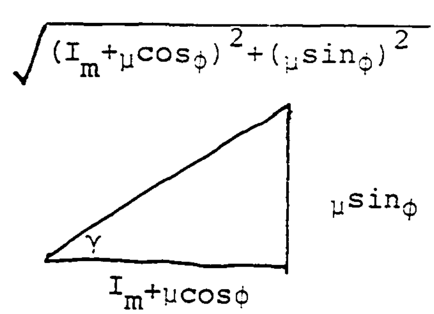
$$= \sqrt{(I_m+\mu\cos\phi)^2+(\mu\sin\phi)^2}\left[\frac{(I_m+\mu\cos\phi)\sin \omega t}{\sqrt{(I_m+\mu\cos\phi)^2+(\mu\sin\phi)^2}}\right.$$

$$\left. + \frac{\mu\sin\phi \quad \cos wt}{\sqrt{(I_m+\mu\cos\phi)^2+(\mu\sin\phi)^2}}\right]$$

$$= \sqrt{(I_m+\mu\cos\phi)^2+(\mu\sin\phi)^2}\,[\sin wt \cos\gamma + \cos wt \sin\gamma]$$

where

$$\gamma = \tan^{-1} \frac{\mu\sin\phi}{I_m+\mu\cos\phi}$$



Since

$$\mu << I_m$$

$$\therefore \sqrt{(I_m+\mu\cos\phi)^2 + (\mu\sin\phi)^2} \simeq I_m+\mu\cos\phi$$

and

$$\tan^{-1} \frac{\mu\sin\phi}{I_m+\mu\cos\phi} \simeq \frac{\mu}{I_m} \sin \phi \qquad (4.27)$$

Hence, the input to nonlinearity can be written as:

$$i(t) = (I_m+\mu\cos\phi)\sin(wt + \frac{\mu}{I_m} \sin\phi) \qquad (4.28)$$

Similarly, if we represent the Describing Function as a complex gain of the transformer input in the series circuit block diagram of Figure 20, substitute 4.28 in Equation 12.5, and manipulate, we can derive the Incremental Describing Function of Appendix D equation 12.6:

$$N(I_m,\phi) = N(I_m) + \frac{I_m\dot{N}(I_m)}{2} (1+e^{-2j\phi})$$

F. Mathematical Model of Ferroresonant Region

1. Pi circuit Stability Curve

A stability curve based on the two-slope incremental input Describing Function is derived.

The Two-slope Incremental Input Describing Function is represented in the block diagram to study jump resonance of a nonlinear system:

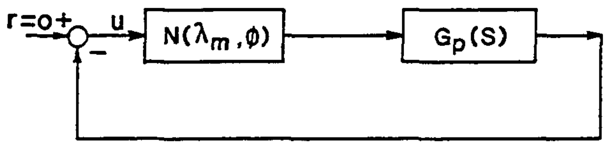


Figure 21. Stability block diagram for the pi circuit

$$r - UNG = U$$

$$r = U + UNG$$

$$r = U(1+NG)$$

If we assume that the oscillations are sustained without input signal, then

$$r = 0$$

and

$$0 = 1+NG$$

or

$$0 = 1+N(\lambda_m,\phi)\ G(s)$$

$$N(\lambda_m,\phi) G_p(s) = -1$$

or

$$G_p(s) = -\frac{1}{N(\lambda_m,\phi)} \qquad (4.29)$$

For a fixed λ_m, the above function in the analysis of the nonlinear system plays the same role of the stability point in the linear system. Therefore, it is called the stability curve. Graphically, this means that the intersection of the left- and right-hand functions indicates the critical points.

Assume $G_p(j\omega) = U(\omega)+jV(\omega)$

Substitute quation 12.3 in 4.29 to obtain:

$$1 + [N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}(1+e^{-j2\phi})][U(\omega)+jV(\omega)] = 0$$

$$1 + [N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}(1+\cos 2\phi - j\sin 2\phi)][U(\omega)+jV(\omega)] = 0$$

$$1 + [N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}(1+\cos 2\phi) - \frac{j\lambda_m \dot{N}(\lambda_m)}{2}\sin 2\phi]$$

$$x\ [U(\omega)+jV(\omega)] = 0$$

$$1 + [N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}(1+\cos 2\phi) - j\frac{\lambda_m \dot{N}(\lambda_m)}{2}\sin 2\phi]U(\omega)$$

$$+ [N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}(1+\cos 2\phi)$$

$$- j\frac{\lambda_m \dot{N}(\lambda_m)\sin 2\phi}{2}]jV(\omega) = 0$$

$$1 + [N(\lambda_m) + \frac{\lambda_m N(\lambda_m)}{2}(1+\cos 2\phi)]U(\omega)$$

$$+ \frac{\lambda_m \dot{N}(\lambda_m)\sin 2\phi}{2} V(\omega) = 0$$

and

$$\left[N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}(1+\cos 2\phi)\right]V(\omega)$$

$$- \left[\frac{\lambda_m \dot{N}(\lambda_m)^{2}}{2}\sin 2\phi\right]U(\omega) = 0$$

$$1 + \left[N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}\right]U(\omega) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}\cos 2\phi\ U(\omega)$$

$$+ \frac{\lambda_m \dot{N}(\lambda_m)\sin 2\phi}{2}\ V(\omega) = 0$$

and

$$V(\omega)\left[N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}\right] + \frac{\lambda_m \dot{N}(\lambda_m)}{2}\cos 2\phi\ V(\omega)$$

$$- \frac{\lambda_m \dot{N}(\lambda_m)}{2}\ \sin 2\phi\ U(\omega) = 0$$

For simplification assume

$$x = N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}$$

$$y = \frac{\lambda_m \dot{N}(\lambda_m)}{2}$$

$$\therefore\ 1 + xu + y\cos 2\phi u + y\sin 2\phi v = 0 \qquad (4.30a)$$

and

$$vx + y\cos 2\phi v - y\sin 2\phi u = 0 \qquad (4.30b)$$

Multiply equation 4.30a by v and 4.30b by u and subtract:

$$v + xuv + yuv\cos 2\phi + v^2 y \sin 2\phi = 0$$

$$\mp xuv \mp yuv\cos 2\phi \pm yu^2 \sin 2\phi = 0$$

$$v + v^2\, y\sin 2\phi + yu^2 \sin 2\phi = 0$$

$$\sin 2\phi = \frac{-v}{y(v^2+u^2)}$$

Multiply equation 4.30a by u and equation 4.30b by v and add:

$$u + xu^2 + yu\cos 2\phi + yvu \sin 2\phi = 0$$

$$v^2 x + yv^2 \cos 2\phi - yvu \sin 2\phi = 0$$

$$u + xu^2 + v^2 x + y\cos 2\phi[u^2+v^2] = 0$$

$$y\cos 2\phi[u^2+v^2] = -u - xu^2 - v^2 x$$

$$\cos 2\phi = -\frac{u+x(u^2+v^2)}{y(u^2+v^2)}$$

Since $\sin^2 2\phi + \cos^2 2\phi = 1$

$$\therefore [+\frac{u+x(u^2+v^2)}{y(u^2+v^2)}]^2 + [\frac{+v}{y(v^2+u^2)}]^2 = 1$$

$$[u+x(u^2+v^2)]^2 + v^2 = y^2(v^2+u^2)^2$$

$$u^2 + 2ux(u^2+v^2) + (u^2+v^2)^2 x^2 + v^2 = y^2(v^2+u^2)^2$$

$$u^2+2ux(u^2+v^2) + (u^2+v^2)^2x^2 + v^2 = y^2(v^2+u^2)^2$$

Dividing by u^2+v^2

$$1 + 2ux + (u^2+v^2)x^2 = y^2(v^2+u^2)$$

$$1 + 2ux + u^2x^2 + v^2x^2 - y^2v^2 - y^2u^2 = 0$$

$$1 + 2ux + u^2(x^2-y^2) + v^2(x^2-y^2) = 0$$

$$1 + 2ux + u^2\left[\left(N+\frac{\lambda_m\dot{N}}{2}\right)^2-\left(\frac{\lambda_m\dot{N}}{2}\right)^2\right] + v^2\left[\left(N+\frac{\lambda_m\dot{N}}{2}\right)^2 - \left(\frac{\lambda_m\dot{N}}{2}\right)^2\right] = 0$$

$$1 + 2u\left(N+\frac{\lambda_m\dot{N}}{2}\right) + u^2\left[N^2+N\lambda_m\dot{N} + \left(\frac{\lambda_m\dot{N}}{2}\right)^2-\left(\frac{\lambda_m\dot{N}}{2}\right)^2\right]$$

$$+ v^2\left[N^2+N\lambda_m\dot{N} + \left(\frac{\lambda_m\dot{N}}{2}\right)-\left(\frac{\lambda_m\dot{N}}{2}\right)^2\right] = 0$$

$$1 + 2u\left(N+\frac{\lambda_m\dot{N}}{2}\right) + u^2[N^2+N\lambda_m\dot{N}] + v^2[N^2+N\lambda_m\dot{N}] = 0$$

$$\frac{1}{N^2+N\lambda_m\dot{N}} + \frac{2u\left(N+\frac{\lambda_m\dot{N}}{2}\right)}{N^2+N\lambda_m\dot{N}} + u^2 + v^2 = 0$$

$$u^2 + 2u\left(\frac{N+\frac{\lambda_m\dot{N}}{2}}{N^2+N\lambda_m\dot{N}}\right) + \left(\frac{N+\frac{\lambda_m\dot{N}}{2}}{N^2+N\lambda_m\dot{N}}\right)^2 + v^2 = \left(\frac{N+\frac{m\dot{N}}{2}}{N^2+N\lambda_m\dot{N}}\right)^2 - \frac{1}{N^2+N\lambda_m\dot{N}}$$

$$\left[u + \frac{N+\frac{\lambda_m\dot{N}}{2}}{N^2+N\lambda_m\dot{N}}\right]^2+v^2 = \frac{\left(N+\frac{\lambda_m\dot{N}}{2}\right)^2}{(N^2+N\lambda_m\dot{N})^2} - \frac{1}{N^2+N\lambda_m\dot{N}}$$

$$[u + \frac{2N+\lambda_m\dot{N}}{2(N^2+N\lambda_m\dot{N})}]^2+v^2 = \frac{N^2+N\lambda_m\dot{N} + \frac{\lambda_m^2\dot{N}^2}{4} - N^2 - N\lambda_m\dot{N}}{(N^2+N\lambda_m\dot{N})^2}$$

$$[u + \frac{2N+\lambda_m\dot{N}}{2(N^2+N\lambda_m\dot{N})}]^2+v^2 = [\frac{\lambda_m\dot{N}}{2(N^2+N\lambda_m\dot{N})}]^2 \qquad (4.31)$$

Graphically interpreted, the above equation represents a family of circles of constant λ_m with centers at $[-\frac{2N+\lambda_m\dot{N}}{2(N^2+N\lambda_m\dot{N})}, 0]$ and radii of $\frac{\lambda_m\dot{N}}{2(N^2+N\lambda_m\dot{N})}$. These circles that represent the Stability Curve combined with the linear part of the pi system representation [G(s)] define whether or not ferroresonance will occur in the given pi-circuit configuration.

2. Series circuit Stability Curve

Similarly, the Incremental Input Describing Function of the series system is represented by the block diagram in the figure below:

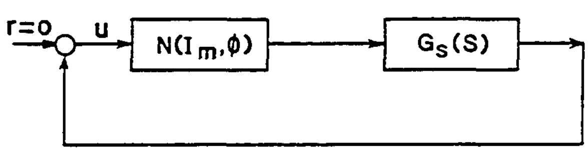


Figure 22. Stability block diagram for the series circuit

$$r - UN(I_m, \phi)G = U$$

$$r = U + UNG$$

$$r = U(1+NG)$$

$$0 = 1+NG$$

$$NG = -1$$

$$\therefore N(I_m, \phi) = -\frac{1}{G_s(S)} \tag{4.32}$$

Graphically, this means that the intersection of the left- and right-hand side functions indicate the critical points. For a fixed I_m, the function $-1/N(I_m, \phi)$ plays the role of stability point in the linear system. In the non-linear system, it is the stability curve since it can take many values when ϕ changes from 0 to 2π. Similarly, if $G_s(j\omega) = u(\omega) + jv(\omega)$ and if we substitute equation 12.6 in 4.32, it can be shown that:

$$[U + \frac{2N+I_m\dot{N}}{2(N^2+NI_m\dot{N})}]^2 + V^2 = [\frac{I_m\dot{N}}{2(N^2+NI_m\dot{N})}]^2 \tag{4.33}$$

Geometrically interpreted, the above equation represents a family of constant I_m circles with centers at $[\frac{2N+I_m\dot{N}}{2(N^2+NI_m\dot{N})}, 0]$ and radii of $[\frac{I_m\dot{N}}{2(N^2+NI_m\dot{N})}]$. These circles, which represent the Stability Curve combined with the Linear Part of the series system representation $[G_s(S)]$, define whether or not ferroresonance will occur in the given series circuit configuration.

V. ANALYSIS OF FERRORESONANCE

A. Prediction of Ferroresonance

In general, a jump (instability) is indicated when the locus of the linear system, G(1j,R,L,C), on the complex plane passes through any portion of the stability curve.

However, to determine whether ferroresonance occurs in a particular power system that includes only one transformer, its circuit must be reduced to either pi or series circuit configuration. The linear portion of the circuit could be replaced with its Thevenins equivalent expanded into a form similar to G(s).

1. Pi circuit

At particular amplitude of the input voltage, if the 60 Hertz point location on the Frequency Response, G(s), is inside the stability curve, fundamental ferroresonance will occur in the form of a jump from the lower to higher lambda intersecting circles where their point of intersection lies on power system frequency point.

2. Series circuit

At any particular amplitude of the input current, if the 60 Hertz point location on the Frequency Response, G(s), is inside the stability curve, fundamental ferroresonance will occur in the form of a jump from the lower to higher current intersecting circles where their point of intersection lies on power system frequency point.

B. Determination of Critical R, L, and C

To design power circuits free of ferroresonance it is necessary to determine a complete set of the critical parameters R, L, and C that trigger the phenomenon.

1. Pi circuit

For this analysis, the real and imaginary parts of the 60 Hz point response function G(1j,R,L,C) are derived from equation 4.18:

$$G_p(s) = \frac{R+SL}{S(S^2LC+SRC+1)}$$

$$G_p(s) = \frac{R+jWL}{jW(j^2W^2LC+jWRC+1)}$$

$$= \frac{R+jWL}{jW(-W^2LC+jWRC+1)}$$

$$= \frac{R+jWL}{jW(1-W^2LC)-W^2RC}$$

$$= \frac{-(R+jWL)}{W^2RC-jW(1-W^2LC)}$$

$$= \frac{-(\frac{R}{\omega} + jL)}{WRC-j(1-W^2LC)}$$

$$= \frac{-(\frac{R}{\omega}+jL)\,[WRC+j(1-W^2LC)]}{[WRC-j(1-W^2LC)]\,[WRC+j(1-W^2LC)]}$$

$$= \frac{-WRC(\frac{R}{W}+jL)-j(\frac{R}{W}+jL)(1-W^2LC)}{(WRC)^2+(1-W^2LC)^2}$$

$$= \frac{-R^2C-jWLRC-j[(\frac{R}{W}+jL)-W^2LC(\frac{R}{W}+jL)]}{W^2R^2C^2+(1-W^2LC)^2}$$

$$= \frac{-R^2C-jWLRC-j[\frac{R}{W}+jL-WLCR-jW^2L^2C]}{W^2R^2C^2+(1-W^2LC)^2}$$

$$G_p(jW) = \frac{-R^2C-jWLRC-j\frac{R}{W}+L+jWLCR-W^2L^2C}{W^2R^2C^2+(1-W^2LC)^2}$$

$$G_p(j1) = \frac{-R^2C+L-L^2C-jR}{R^2C^2+(1-LC)^2}$$

or

$$Re[G_p(j1,R,L,C)] = \frac{-R^2C+L-L^2C}{R^2C^2+(1-LC)^2} \tag{5.1}$$

and

$$I_{mag}[G_p(j1,R,L,C)] = \frac{-R}{R^2C^2+(1-LC)^2} \tag{5.2}$$

It appears that while the stability curve representing the nonlinear region is independent of the parameters R, L, and C, the locus of the 60 Hz response function

G(1j,R,L,C) is dependent on them.

Therefore, by changing the values of R, L, and C, it is possible to move the 60 Hz point on the complex plane in the desired direction where ferroresonance will not occur in the system.

However, for the power system, it is more convenient to express R and L parameters of G(1j,R,L,C) as constants and vary only the capacitance. The corresponding change in G(1j,R,L,C) when C changes is the capacitance response (G1j,R,L,C).

For instance, in the pi circuit configuration of Figure 1, the resistance 9.02 Ω, inductance 316 mH, and capacitance 6 to 180 μF were assumed, i.e., the capacitance value of 6 μF was incremented in steps of 1 μF at a time up to 180 μF. For each capacitance value, the capacitance response of the 60 Hz point G(1j,R,L,C) was calculated and plotted on a common complex plane with the stability curve representing the nonlinear region. As shown on Figure 23, it appears that there exists a range of capacitance values of (23.5-42 μF) that could place the 60 Hz point locus inside the nonlinear region and eventually cause ferroresonance in the system. Therefore, the critical parameters are:

R (Ω)	L (mH)	C (μF)
9.02	316	23.5-42

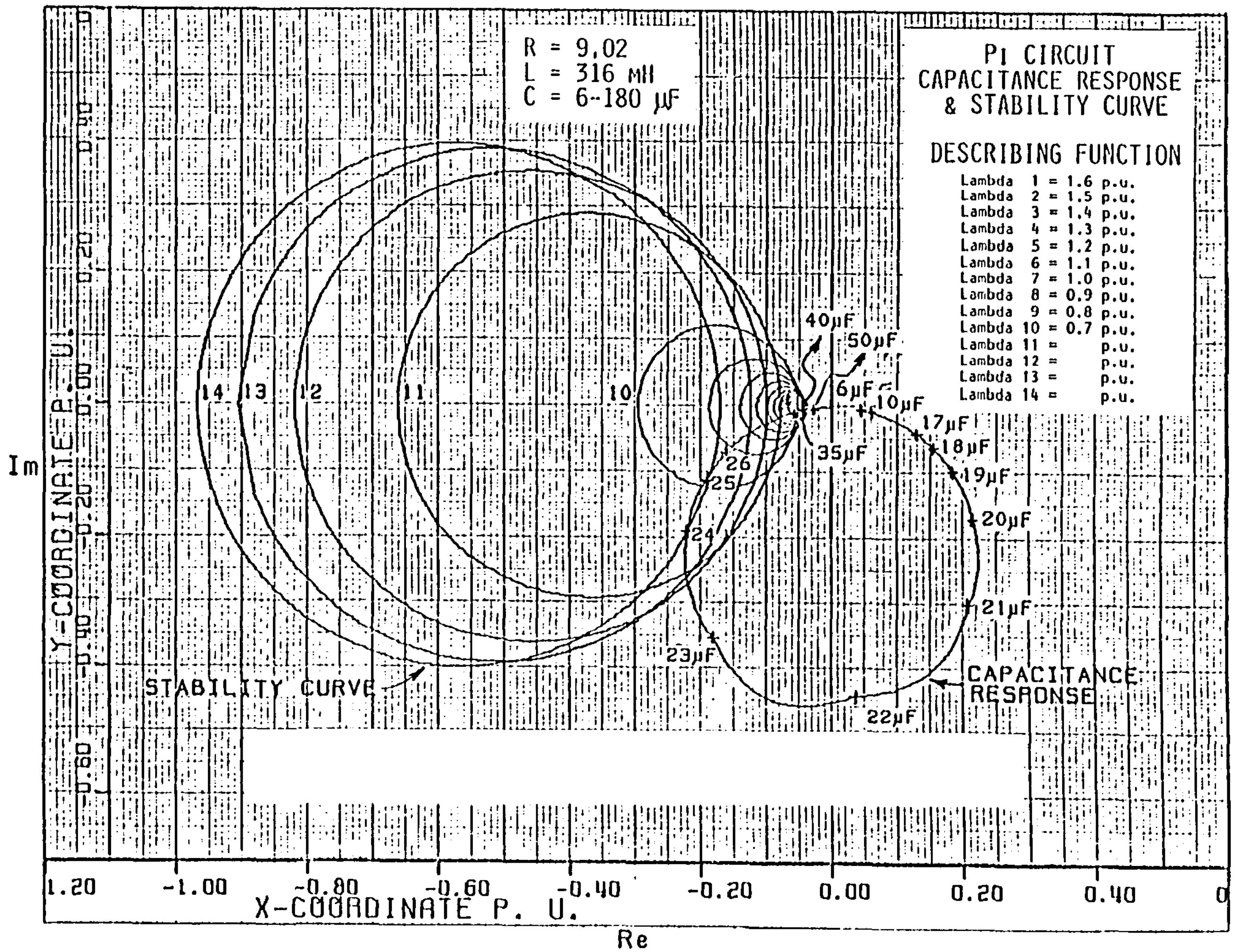


Figure 23. Stability and capacitance response curves for the pi circuit

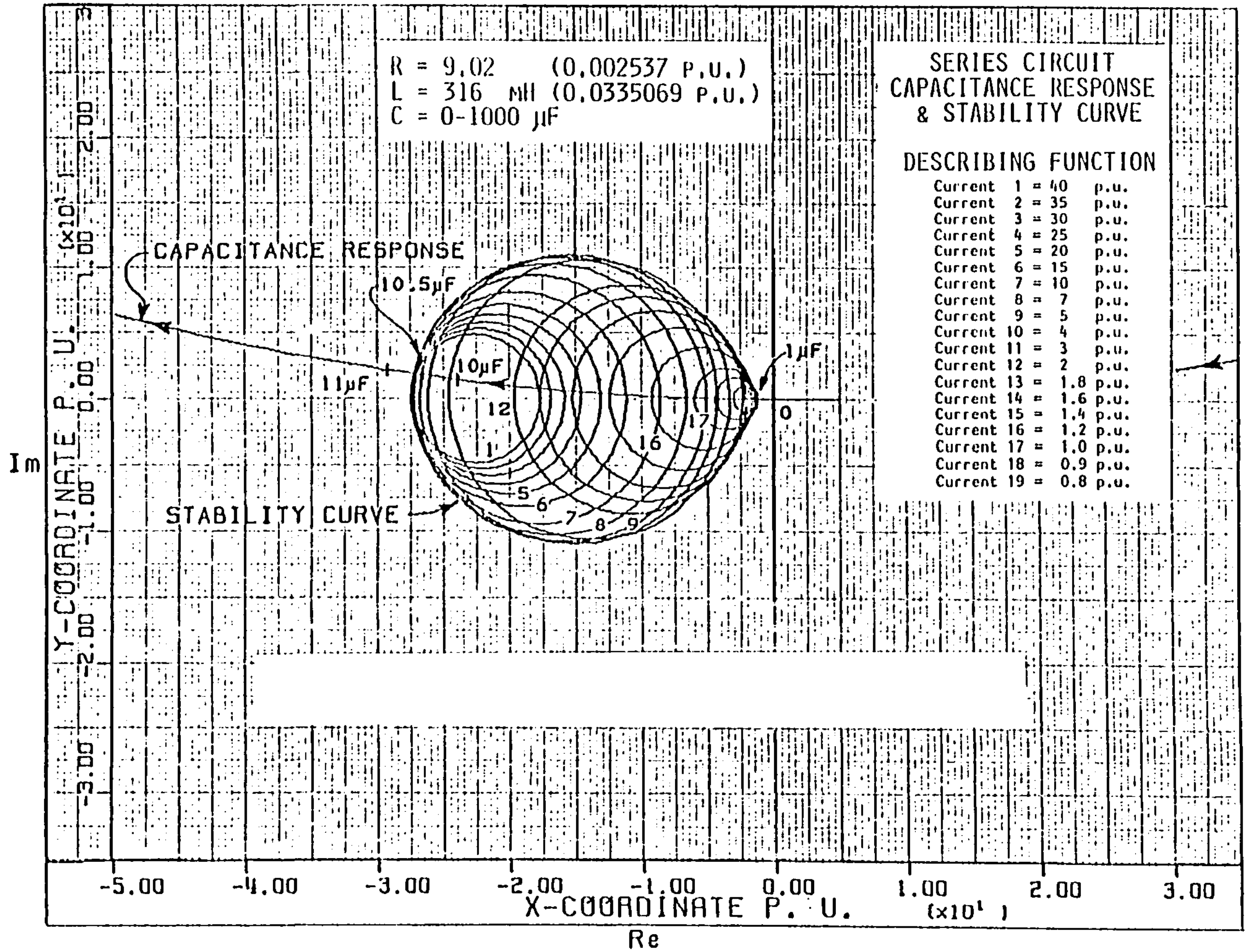


Figure 24. Stability and capacitance response curves of the series circuit

It appears that there exists two capacitance threshold values, the upper and the lower limits.

2. Series circuit

It can be shown that the real and imaginary parts of the 60 Hz point response function G(1j,R,L,C) are derived from equation 4.23:

$$G_s(S) = \frac{CS^2}{1+RCS+LCS^2}$$

$$G(jw) = \frac{Cj^2W^2}{1+jRCW+j^2LCW^2}$$

$$\therefore\ G_s(j1) = \frac{-C}{(1-LC) + jRC}$$

or

$$Re[G_s(j1,R,L,C)] = \frac{LC^2-C}{R^2C^2+(1-LC)^2} \qquad (5.3)$$

and

$$I_{mag}[G_s(j1,R,L,C)] = \frac{RC^2}{R^2C^2+(1-LC)^2} \qquad (5.4)$$

In the series circuit configuration, the resistance 9.02 Ω, inductance 316 mH, and capacitance 0 to 1000 μF were assumed, i.e., a capacitance value of 1 μF, was incremented in steps of 1 μF at a time up to 1000 μF. For each capacitance value in the circuit, the corresponding value of the capacitance response G(1j,R,L,C) is calculated and plotted on a common complex plane with the stability curve.

As shown in Figure 24, it appears that there exists a range of capacitance values of 1 to 10.5 μF that could place the 60 Hz point locus inside the stability curve and eventually cause ferroresonance in the system. Therefore, the critical parameters are:

R (Ω)	L (mH)	C (μF)
9.02	316	1-10.6

Hence, there exists only an upper limit for the capacitance threshold.

C. Analytical and Graphical Determination of the Critical Flux Linkage λ_m, Current I_m, and Inductance L_{eq}

1. Pi circuit

An expression to determine critical lambdas can be derived if we assume that $-K_1$ and $-K_2$ are the real and imaginary parts of 60 Hz point located either on the envelope or inside the stability curve:

$$[-K_1 + \frac{2N+\lambda\dot{N}}{2(N^2+N\lambda\dot{N})}]^2 + [-K_2]^2 = [\frac{\lambda\dot{N}}{2(N^2+N\lambda\dot{N})}]^2$$

$$[-K_1 + \frac{N+\lambda\dot{N}}{2N(N+\lambda\dot{N})} + \frac{N}{2N(N+\lambda\dot{N})}]^2 = [\frac{N+\lambda\dot{N}}{2N(N+\lambda\dot{N})} - \frac{N}{2N(N+\lambda\dot{N})}]^2$$

$$[-K_1 + \frac{1}{2N} + \frac{1}{2(N+\lambda\dot{N})}]^2 + K_2^2 = [\frac{1}{2N} - \frac{1}{2(N+\lambda\dot{N})}]^2$$

$$[\frac{1}{2N(N+\lambda\dot{N})}]^2[2N(N+\lambda\dot{N})(-K_1) + (N+\lambda\dot{N})+N]^2 + K_2^2$$

$$= [\frac{1}{2N(N+\lambda\dot{N})}]^2[(N+\lambda\dot{N}) - N]^2$$

$$\frac{1}{4N^2(N+\lambda\dot{N})}[2N(N+\lambda\dot{N})(-K_1)+2N+\lambda\dot{N}]^2 + K_2^2 = \frac{1}{4N^2(N+\lambda\dot{N})^2}[\lambda\dot{N}]^2$$

$$[2N(N+\lambda\dot{N})(-K_1) + (2N+\lambda\dot{N})]^2 + 4K_2^2N^2(N+\lambda\dot{N})^2 = (\lambda\dot{N})^2$$

where

$$N = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} + \frac{\lambda_0}{\lambda_m}\sqrt{1-(\frac{\lambda_0}{\lambda_m})^2}] + m_2$$

From IV.C

$$\lambda_m\dot{N} = -\frac{4(m_1-m_2)}{\pi}\frac{\lambda_0}{\lambda_m}\sqrt{1-(\frac{\lambda_0}{\lambda_m})^2}$$

Assume

$$C = \frac{2(m_1-m_2)}{\pi}$$

$$X = \frac{\lambda_0}{\lambda_m}$$

and

$$N = C\sin^{-1}X + CX\sqrt{1-X^2} + m_2$$

$$\lambda\dot{N} = -2CX\sqrt{1-X^2}$$

$$\therefore [2(C\sin^{-1}X+CX\sqrt{1-X^2}+m_2)(C\sin^{-1}X-CX\sqrt{1-X^2}+m_2)(-K_1)$$

$$+ 2C\sin^{-1}X+2m_2]^2$$

$$+ 4K_2^2[C\sin^{-1}X+CX\sqrt{1-X^2}+m_2]^2[C\sin^{-1}X-CX\sqrt{1-X^2}+m_2]^2$$

$$= 4C^2X^2(1-X^2)$$

$$4K_1^2[C\sin^{-1}X+m_2)^2 - (CX\sqrt{1-X^2})^2]^2 - 8K_1[(C\sin^{-1}X+m_2)^2$$

$$- (CX\sqrt{1-X^2})^2](C\sin^{-1}X+m_2)$$

$$+ (2C\sin^{-1}X+2m_2)^2 + 4K_2^2[(\sin^{-1}X+m_2)^2 - (CX\sqrt{1-X^2})^2]^2$$

$$= 4C^2X^2(1-X^2)$$

$$[(C\sin^{-1}X+m_2)^2-(CX\sqrt{1-X^2})^2]^2(4K_1^2+4K_2^2)$$

$$+ [(C\sin^{-1}X+m_2)^2 - (CX\sqrt{1-X^2})^2][-8K_1(C\sin^{-1}X+m_2)+4] = 0$$

$$[(C\sin^{-1}X+m_2)^2-(CX\sqrt{1-X^2})^2][\{(C\sin^{-1}X+m_2)^2$$

$$- (CX\sqrt{1-X^2})^2\}(4K_1^2+4K_2^2)$$

$$+ (-8K_1C\sin^{-1}X-8K_1m_2+4)] = 0$$

either

$$\{(C\sin^{-1}X+m_2)^2 - (CX\sqrt{1-X^2})^2\}(4K_1^2+4K_2^2)$$

$$- 8K_1C\sin^{-1}X-8K_1m_2+4 = 0$$

or

$$(C\sin^{-1}X+m_2)^2 - (CX\sqrt{1-X^2})^2 = 0$$

Since the 60 Hz point is of interest we solve:

$$\{(C\sin^{-1}X+m_2)^2 - (CX(\sqrt{1-X^2})^2\}(4K_1^2+4K_2^2) - 8K_1C\sin^{-1}X$$

$$- 8K_1m_2+4 = 0$$

$$(C\sin^{-1}X+m_2)^2(4K_1^2+4K_2^2) - (CX\sqrt{1-X^2})^2(4K_1^2+4K_2^2)$$

$$- 8K_1C\sin^{-1}X - 8K_1m_2+4 = 0$$

$$\{C^2(\sin^{-1}X)^2+2Cm_2\sin^{-1}X+m_2^2\}(4K_1^2+4K_2^2)$$

$$- C^2X^2(1-X^2)(4K_1^2+4K_2^2) - 8K_1C\sin^{-1}X - 8K_1m_2+4 = 0$$

$$\{C^2(\sin^{-1}X)^2 + 2Cm_2\sin^{-1}X+m_2^2\}K-C^2X^2(1-X^2)K$$

$$- 8K_1C\sin^{-1}X - 8K_1m_2+4 = 0$$

$$KC^2(\sin^{-1}X)^2 + 2KCm_2\sin^{-1}X + Km_2^2 - 8K_1C\sin^{-1}X$$

$$- 8K_1m_2+4 = C^2KX^2(1-X^2)$$

$$X^2(1-X^2) = \frac{KC^2(\sin^{-1}X)^2 + 2KCm_2\sin^{-1}X + Km_2^2 - 8K_1C\sin^{-1}X - 8K_1m_2+4}{C^2K}$$

$$X^2(1-X^2) = [(\sin^{-1}X)^2 + \frac{2m_2}{C}\sin^{-1}X + \frac{m_2^2}{C^2} - \frac{8K_1}{CK}\sin^{-1}X$$

$$- \frac{8K_1m_2}{C^2K} + \frac{4}{C^2K}]$$

$$X^2(1-X^2) = [(\sin^{-1}X)^2 + (\frac{2m_2}{C} - \frac{8K_1}{CK})\sin^{-1}X$$

$$+ (\frac{Km_2^2-8K_1m_2+4}{C^2K})]$$

$$X^4-X^2+(\sin^{-1}X)^2 + (\frac{2m_2}{C} - \frac{8K_1}{CK})\sin^{-1}X + \frac{Km_2^2-8K_1m_2+4}{C^2K} = 0 \quad (5.5)$$

where

$$K = 4(K_1^2+K_2^2)$$

To make a comparison between the analytical and graphical values of λ_m, a set of critical parameters are chosen to place the 60 Hz point inside the stability curve. Therefore, for values of R = 9.02 Ω, L = 316 mH, and C = 35 μF, coordinates of the 60 Hz point are calculated using equations 5.1 and 5.2:

$$Re[G_p(j1,\ 9.02,\ 316,\ 35)] = K_1 = -0.0570441 \text{ P.U.}$$

$$I_{mag}[G_p(j1,\ 9.02,\ 316,\ 35)] = K_2 = -0.007435452 \text{ P.U.}$$

Critical lambdas and inductances may be obtained from equation 5.5, 4.3 and 4.4 as follows:

$$X^4-X^2 + (\sin^{-1}X)^2 + [\frac{2m_2}{C} - \frac{8K_1}{CK}]\sin^{-1}X + [\frac{Km_2^2-8K_1m_2+4}{C^2K}] = 0$$

where

$$X = \frac{\lambda_0}{\lambda_m}$$

$\lambda_0 = 0.6551724$ P.U.

$m_1 = 1$ P.U.

$m_2 = 27.454377$ P.U.

$K = 4(K_1^2+K_2^2) = 0.0132373$ P.U.

$K_1 = 0.0570441$ P.U.

$K_2 = 0.007435452$ P.U.

$C = \frac{2(m_1-m_2)}{\pi} = -16.8413791$ P.U.

Two solutions are obtained:

$X_1 = 0.8563876$

$X_2 = 0.3243749$

Since

$$X_1 = \frac{\lambda_0}{\lambda m_1}$$

or

$$\lambda_{m_1} = \frac{\lambda_0}{x_1}$$

$$= \frac{0.6551724}{0.8563876}$$

$\lambda_{m_1} = 0.7650419$ P.U., similarly,

$\lambda_{m_2} = 2.0198$

Since

$$L_{eq} = \frac{1}{N(\lambda_m)}$$

and

$$N(\lambda_m) = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{\lambda_0}{\lambda_m} + \frac{\lambda_0}{\lambda_m}\sqrt{[1-(\frac{\lambda_0}{\lambda_m})^2]}] + m_2$$

$N(0.7650419) = 2.6905793$

$$L_1 = \frac{1}{N(\lambda_m)}$$

$$= \frac{1}{N(0.7650419)}$$

$$L_1 = \frac{1}{2.6905793}$$

$L_1 = 0.3716672$

$\therefore$ jump point is $\lambda_{m_1} = 0.7650419$ P.U.

$L_1 = 0.3716672$ P.U.

and

$N(2.0198) = 16.723281$

$$L_2 = \frac{1}{N(2.0198)}$$

$$L_2 = 0.0597969$$

and

$$\lambda_{m_2} = 2.0198 \text{ P.U.}$$
$$L_2 = 0.0597969 \text{ P.U.}$$

Same critical lambdas and inductances may also be obtained from a graph. For values of R = 9.02 Ω, L = 316 mH, and C = 35 μF, the Frequency Response of the Linear System $G_p(s)$ is drawn on a complex plane. Also, for values of m_1 = 1 P.U., m_2 = 27 P.U., and λ_m = 0.8 to 1.7 P.U., the stability curve is drawn on the same complex plane of the Frequency Response.

As shown on Figure 25, two circles pass through the 60 Hz point λ_{m_1} = 0.8 and λ_{m_2} = 1.7. The corresponding critical inductance value of L_1 = 0.29 P.U., at which jump resonance occurred and an inductance of L_2 = 0.075 P.U. are obtained from Figure 26.

As expected, there is a discrepancy between the analytical and the graphical values:

Pi circuit parameters	Analytical λ_m		Graphical λ_m	
	Jump-from	Jump-to	Jump-from	Jump-to
R = 9.02 L = 316 mH C = 35 F	0.77 P.U.	2 P.U.	0.8 P.U.	1.7 P.U.

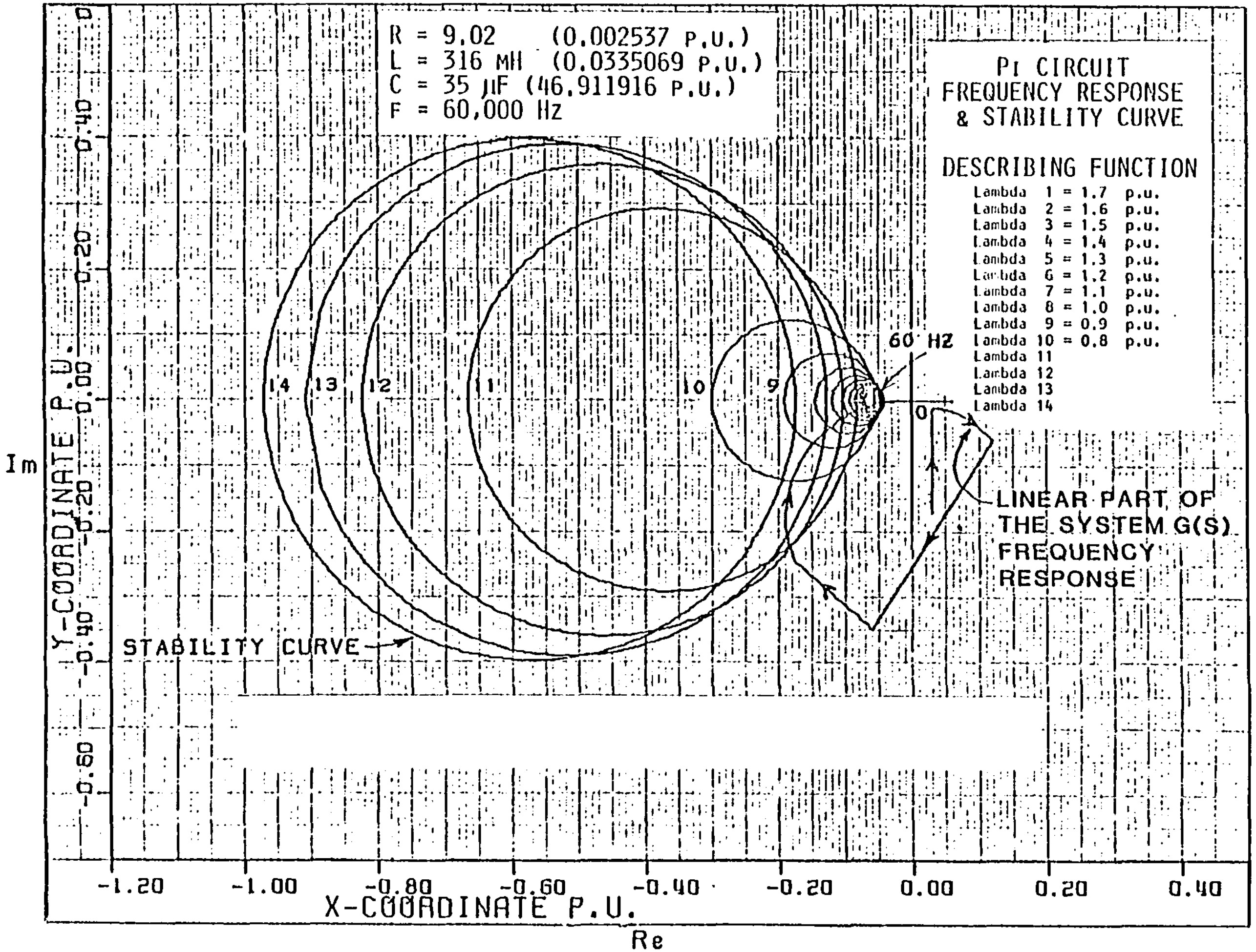


Figure 25. Stability and frequency response curves of the pi circuit of experiment No. 7

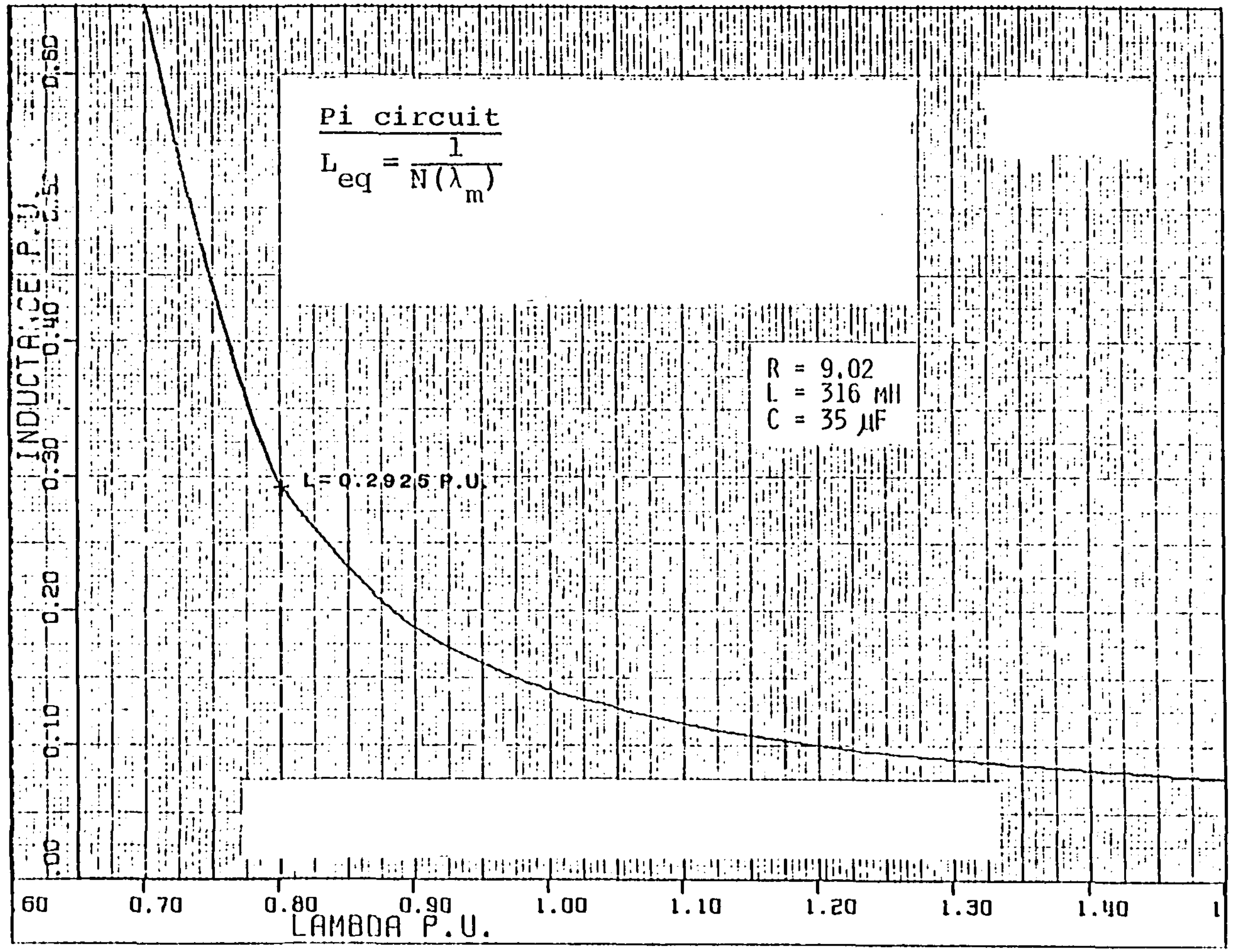


Figure 26. Inductance vs. lambda of the pi circuit of experiment No. 7

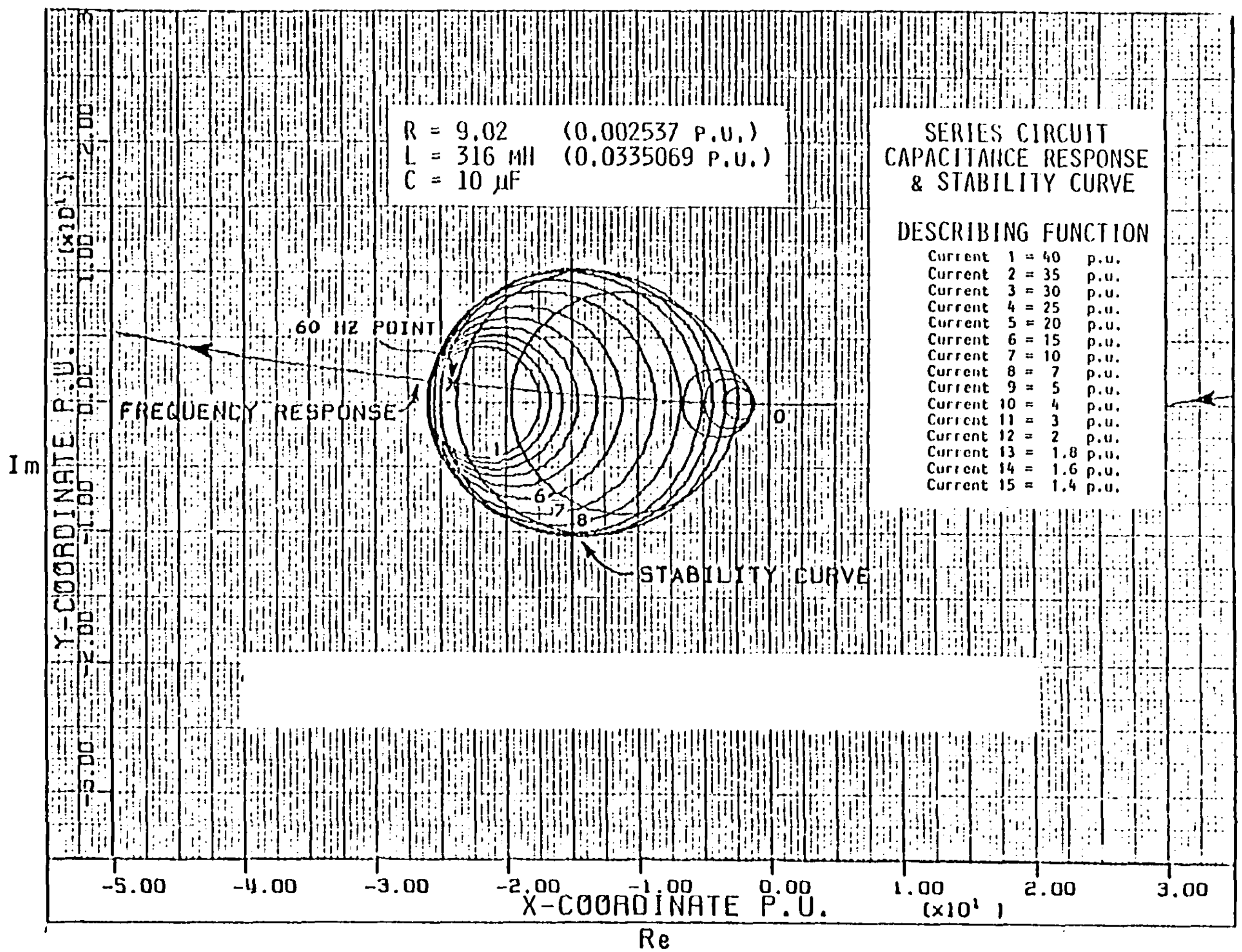


Figure 27. Stability and frequency response curves of the series circuit of experiment No. 6

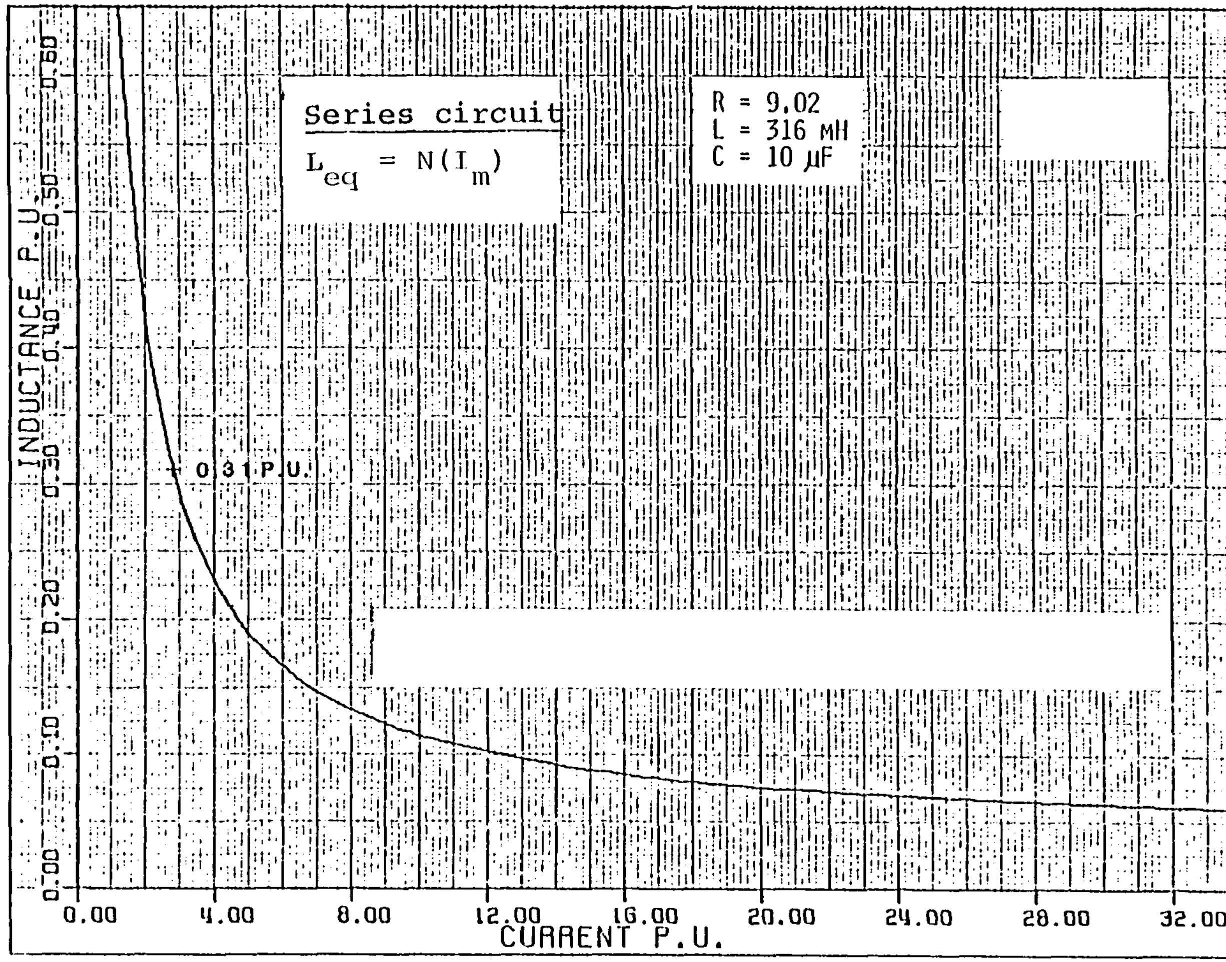


Figure 28. Inductance vs. current of the series circuit experiment No. 6

2. Series circuit

An expression to determine critical currents can be derived if we assume that $-K_3$ and $-K_4$ are the real and imaginary parts of 60 Hz point located either on or inside the stability curve.

Therefore, from equation 4.31:

$$[-K_3 + \frac{2N+I_m\dot{N}}{2(N^2+NI_m\dot{N})}]^2 + (-K_4)^2 = [\frac{I_m\dot{N}}{2(N^2+NI_m\dot{N})}]^2$$

This equation can be rewritten in the following form:

$$x^4-x^2+(\sin^{-1}x)^2 + (\frac{2m_2}{C} - \frac{8K_3}{CK})\sin^{-1}x + \frac{Km_2^2-8K_3m_2+4}{C^2K} = 0 \qquad (5.6)$$

where

$$X = \frac{I}{I_m}$$

$$C = \frac{2(m_1-m_2)}{\pi}$$

$$K = 4(K_3^2 + K_4^2)$$

Similarly, to make a comparison between the analytical and graphical critical currents I_m, a set of critical parameters are chosen to place the 60 Hz point inside the stability curve. Therefore, for values of R = 9.02 Ω, L =

316 mH and C = 10 μF, the coordinates of the 60 Hz point are calculated using equations 5.3 and 5.4:

$$\mathrm{Re}[G_s(j1,\ 9.02,\ 316,\ 10)] = K_3 = -24.23795 \text{ P.U.}$$

$$I_{mag}[G_s(j1,\ 9.02,\ 316,\ 10)] = K_4 = 1.496109 \text{ P.U.}$$

where:

$$X = \frac{\lambda_0}{\lambda_m}$$

$$I_0 = 0.6504065 \text{ P.U.}$$

$$m_1 = 1 \text{ P.U.}$$

$$m_2 = 0.0364241 \text{ P.U.}$$

$$C_1 = \frac{2(m_1 - m_2)}{\pi}$$

$$= 0.6134315$$

$$K = 4(K_3^2 + K_4^2)$$

$$K_3 = -24.23795 \text{ P.U.}$$

$$K_4 = 1.496109 \text{ P.U.}$$

$$\therefore K = 4[(24.23)^2 + (1.496109)^2]$$

$$K = 2357.325$$

Critical currents and inductances may be obtained by using equation 5.6, 4.11, and 4.12 as follows:

$$x^4-x^2 + (\sin^{-1}x)^2 + (\frac{2m_2}{C_1} - \frac{8K_3}{C_1K})\sin^{-1}x$$

$$+ \frac{(Km_2^2 - 8K_3m_2 + 4)}{C_1^2K} = 0$$

$$x^4-x^2 + (\sin^{-1}x)^2 + 0.015336 \sin^{-1}x + 0.000073 = 0$$

and

$$x_1 = 0.2242585$$

since

$$Im_1 = \frac{0.6504065}{0.2242585}$$

$$Im_1 = 2.9002535$$

and

$$x_2 = 0.00476$$

$$\therefore Im_2 = \frac{0.6504065}{0.00476}$$

$$Im_2 = 136.64002$$

Using equations 4.11 and 4.12

$$N(I_m) = \frac{2(m_1-m_2)}{\pi}[\sin^{-1}\frac{I_0}{I_m} + \frac{I_0}{I_0}\sqrt{1-(\frac{I_0}{I_m})^2}] + m_2$$

where

$$m_1 = 1 \text{ P.U.}$$

$$m_2 = 0.0364241 \text{ P.U.}$$

$$C_1 = \frac{2(m_1-m_2)}{\pi}$$

$$C_1 = 0.6134315 \text{ P.U.}$$

$I_0 = 0.6504065$

For $Im_1 = 2.9002535$

$N(Im_1) = 0.3092347$

and

$L = N(I_m)$

$\therefore L_1 = 0.3092347$

Similarly, for

$Im_2 = 136.64002$

$N(Im_2) = 0.0422639 = L_2$

Therefore,

$\therefore Im_1 = 2.9002535$ P.U.

$L_1 = 0.3092347$ P.U.

and

$Im_2 = 136.64002$ P.U.

$L_2 = 0.0422639$ P.U.

Same critical currents and inductances may also be obtained from a graph. For values of R = 9.02 Ω, L = 316 mH, and 10 μF, the Frequency Response of the Linear System $G_s(S)$ is drawn on the complex plane. Also, for values of m_1 = 1 P.U., m_2 = 0.0364241 P.U., and I_m = 1.4 to 140 P.U., the stability curve is drawn on the same complex plane of the Frequency Response.

As shown in Figure 27, two circles pass through the 60 Hz point $I_{m_1} = 5$ P.U. and $I_{m_2} = 130$ P.U. The corresponding critical inductance value of $L_1 = 0.3$ P.U. at which jump resonance occurred and inductance of $L_2 = 0.06$ P.U. are obtained from Figure 28.

As expected, there is a discrepancy between the analytical and the graphical values:

Pi circuit parameters	Analytical I_m		Graphical I_m	
	Jump-from	Jump-to	Jump-from	Jump-to
R = 9.02 L = 316 mH C = 10 F	2.9 P.U.	137 P.U.	5 P.U.	130 P.U.

D. 60 Hertz Locus Sensitivity of Capacitance Response $G_p(1j,R,L,C)$ to Variation in L and R

1. Pi circuit

To measure the sensitivity of the 60 Hz point locus of the capacitance response, G(1j,R,L,C), to variation in L or R, each linear parameter was changed by 10% at a time and its resulting critical capacitance range was calculated as shown:

Parameter	Experimental values	+10% L	-10% L	+10 R	-10% R
L (mH)	316	347.6	284	316	316
R (Ω)	9.02	9.02	9.02	9.92	8.118
C (μF)	23.4-42	21.6-40	26.3-42	23.8-42	23.8-42

If we examine carefully the following graphs, we will observe that the end points of the capacitance response are the only fixed points. In addition, increasing circuits resistance or decreasing its inductance moves the locus of the capacitance response $G_p(1j,R,L,C)$ away from the real axis and decreases the susceptibility of the circuit to ferroresonance and vice versa.

2. Series circuit

Similarly, to measure the sensitivity of the 60 Hz point locus capacitance response G(1j,R,L,C) to variation in L or R, each linear parameter was changed by 10% at a time and its resulting critical capacitance range was calculated as shown:

Parameter	Experimental values	+10% L	-10% L	+10% R	-10% R
L (mH)	316	347.6	284.4	316	316
R (Ω)	9.02	9.02	9.02	9.92	8.118
C (μF)	1-10.6	1-10.2	1-11.1	1-10.6	1-10.6

As opposed to the graphs of the pi circuit, the following graphs reveal that the upper limit for the capacitance values that cause ferroresonance in the series circuit is shown to be more sensitive to variation in inductance than resistance.

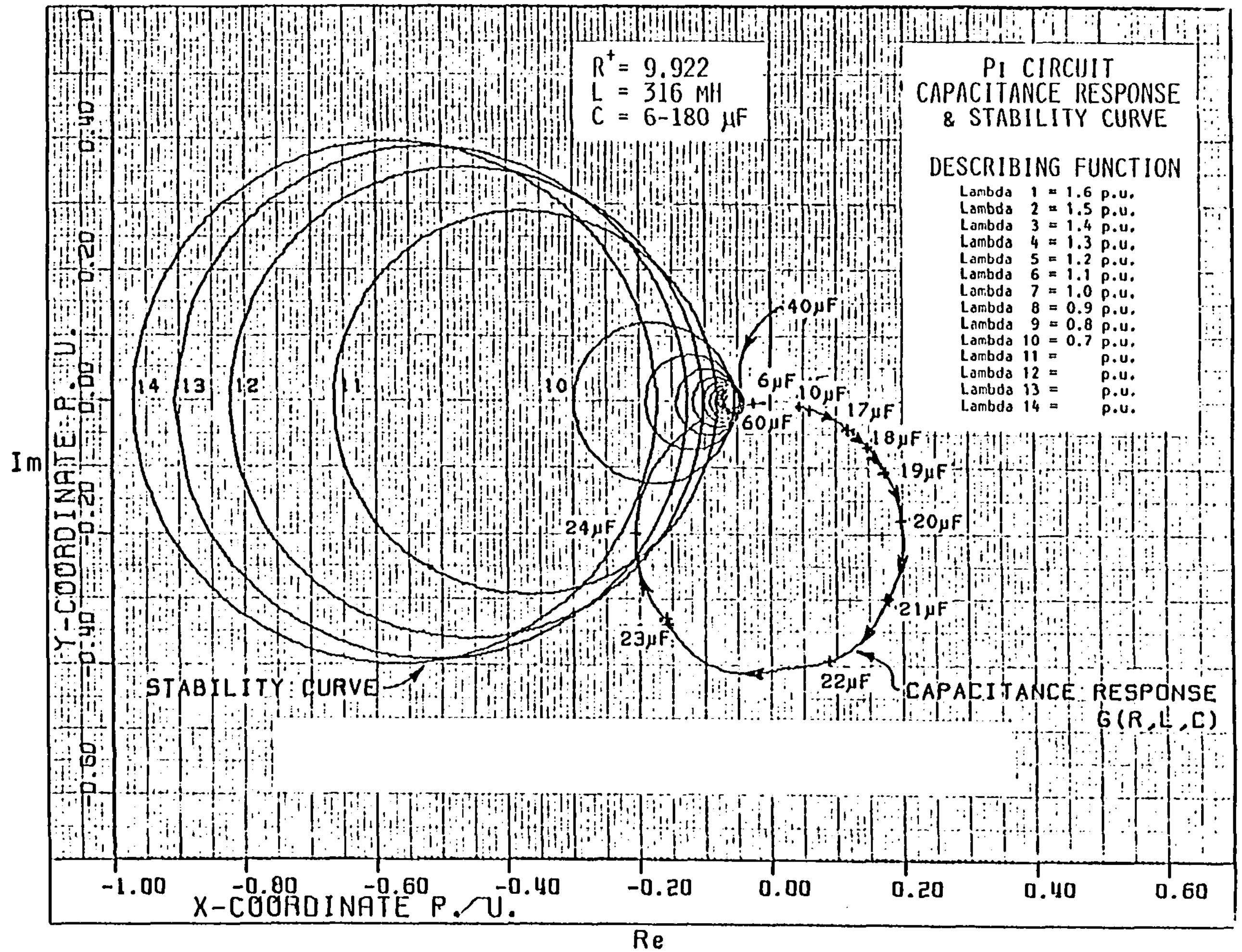


Figure 29. Sensitivity of the capacitance response to increase in resistance of the pi circuit

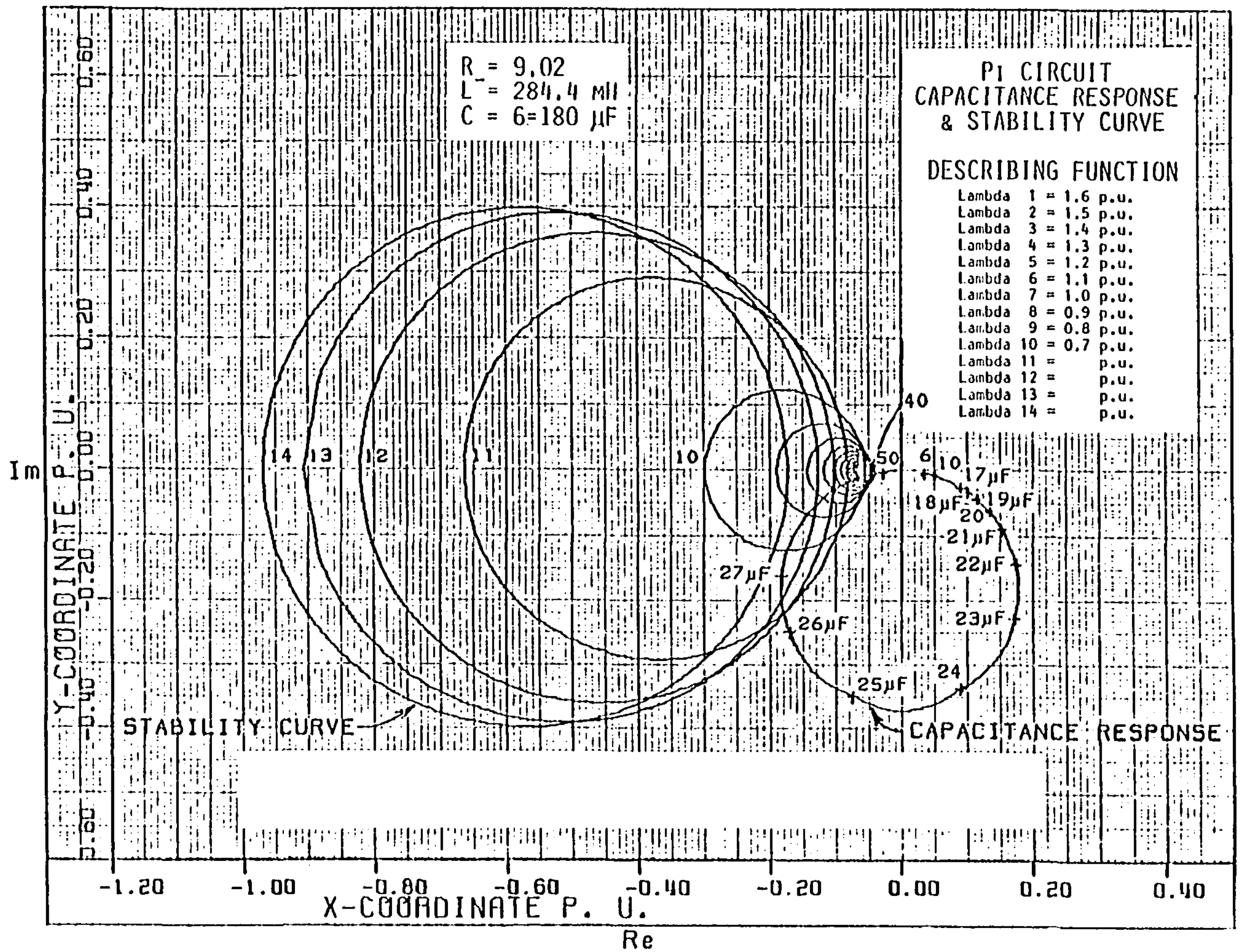


Figure 30. Sensitivity of the capacitance response to decrease in linear inductance of the pi circuit

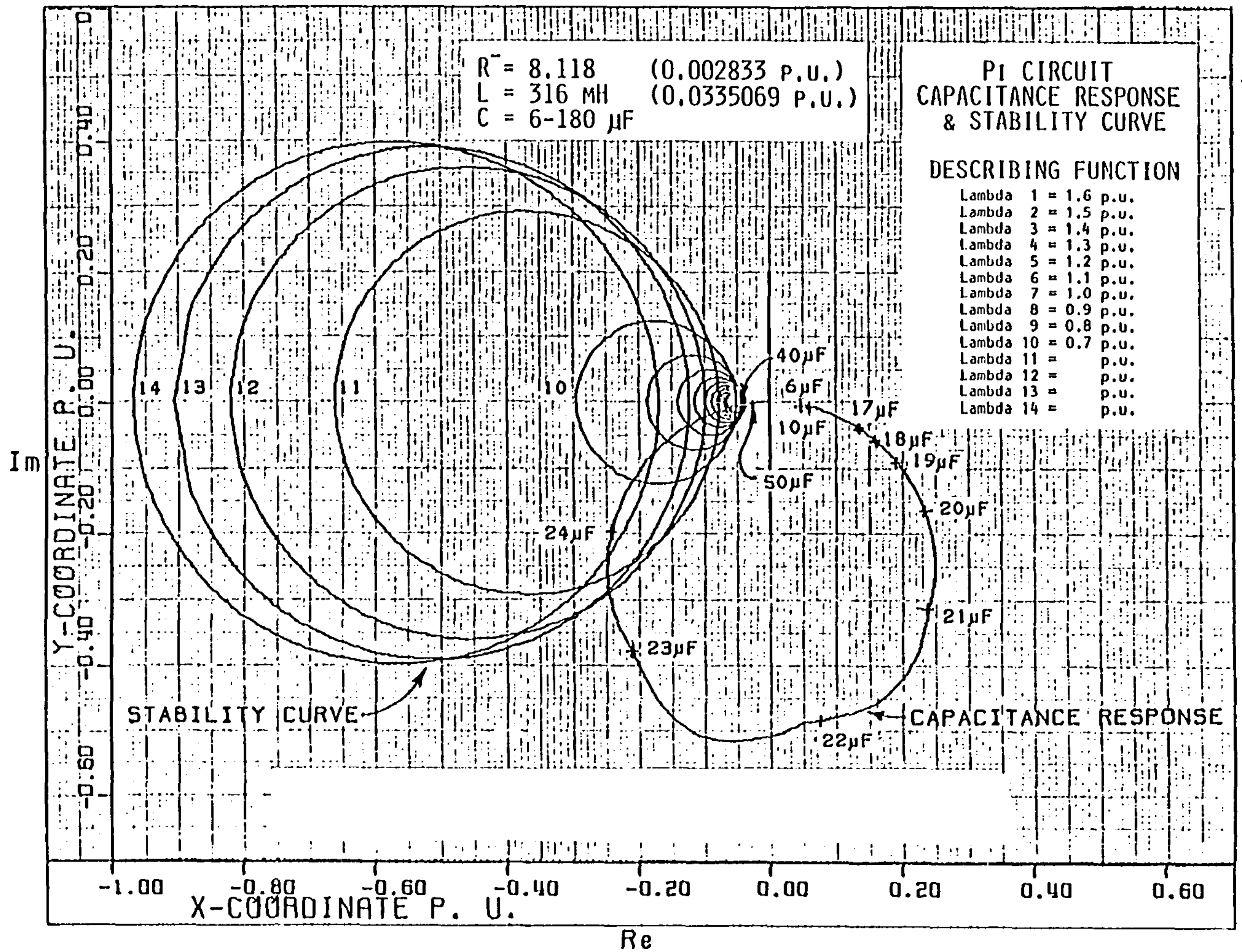


Figure 31. Sensitivity of the capacitance response to decrease in resistance of the pi circuit

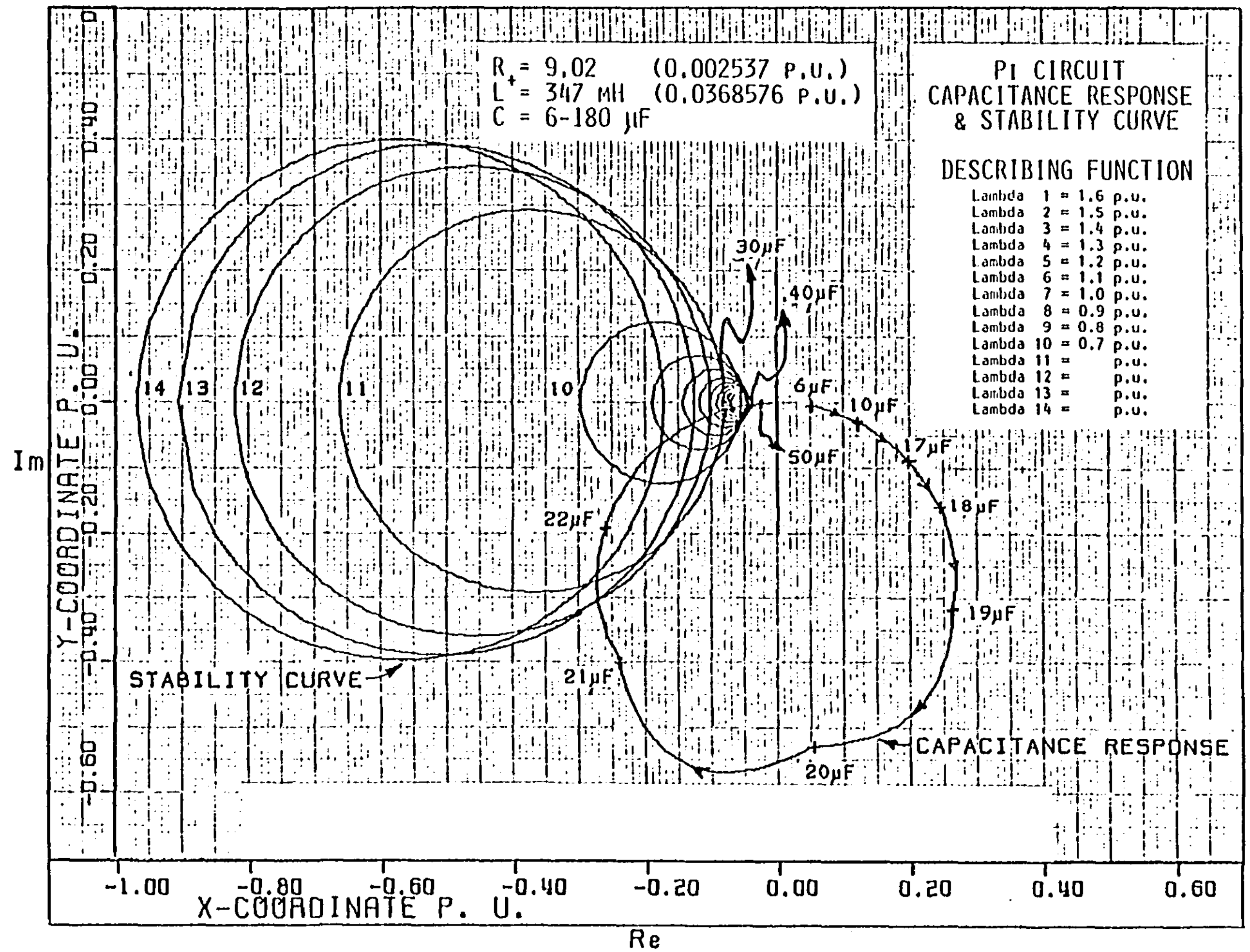


Figure 32. Sensitivity of the capacitance response to increase in linear inductance of the pi circuit

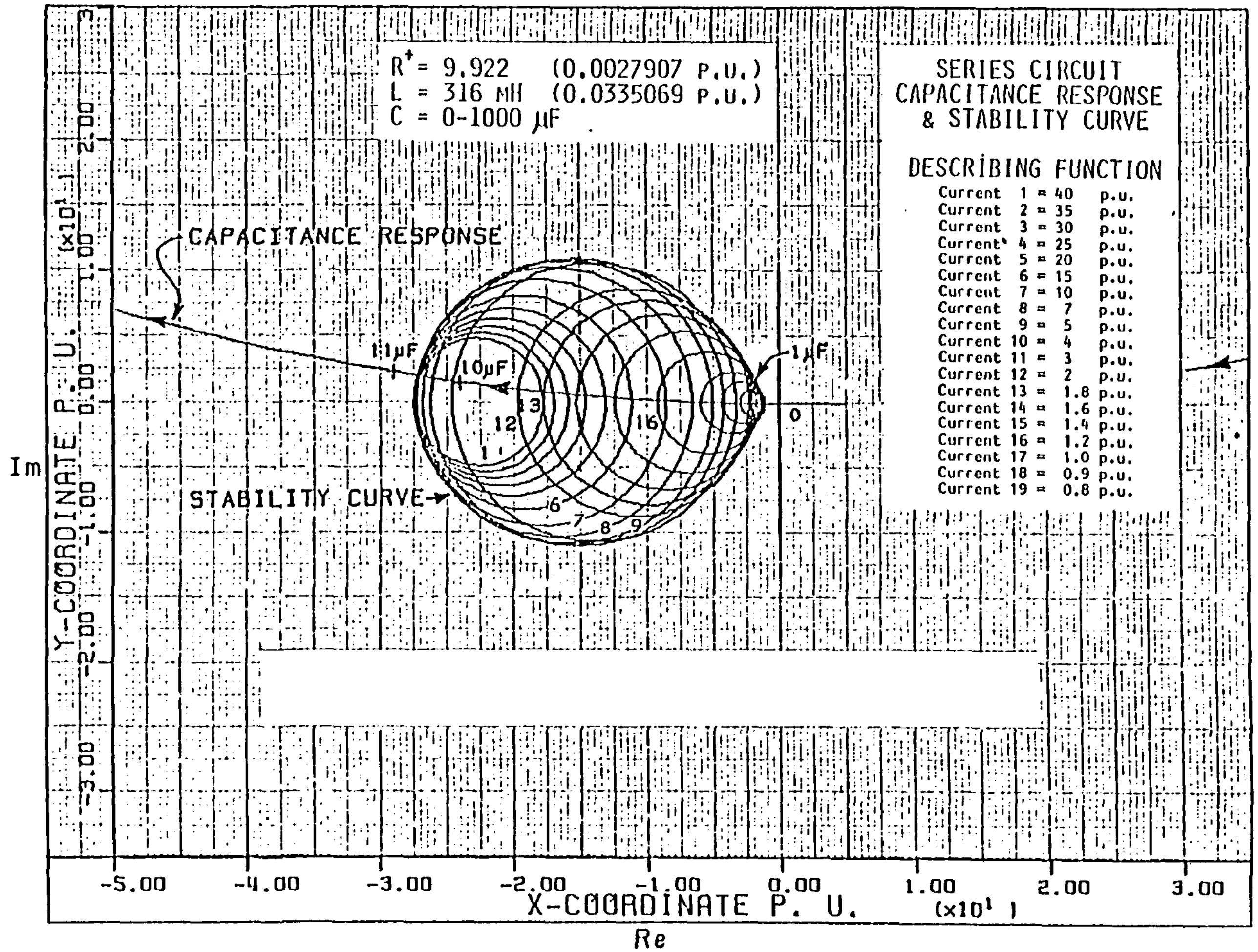


Figure 33. Sensitivity of capacitance response curve to increase in resistance of the series circuit

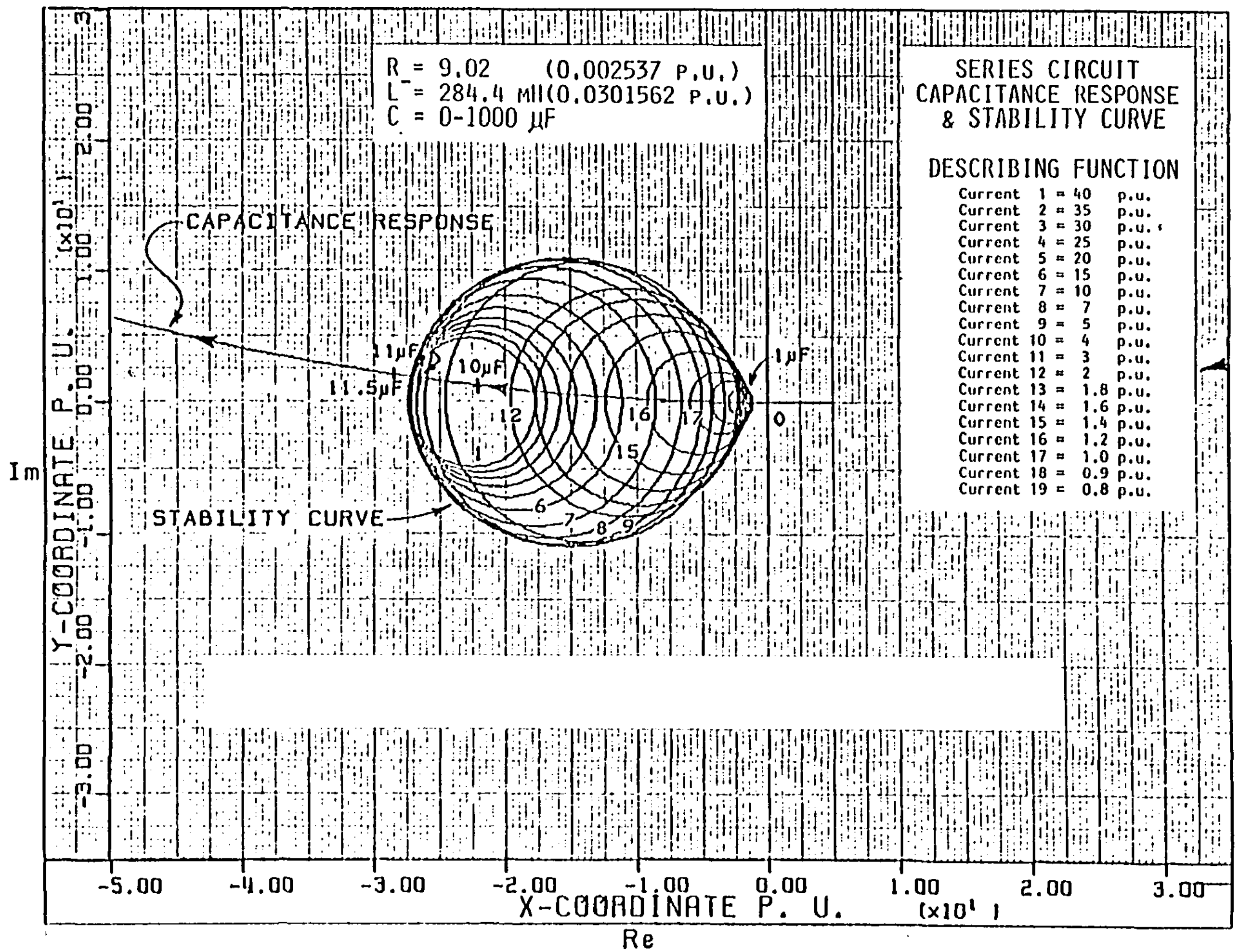


Figure 34. Sensitivity of capacitance response curve to decrease in linear inductance of the series circuit

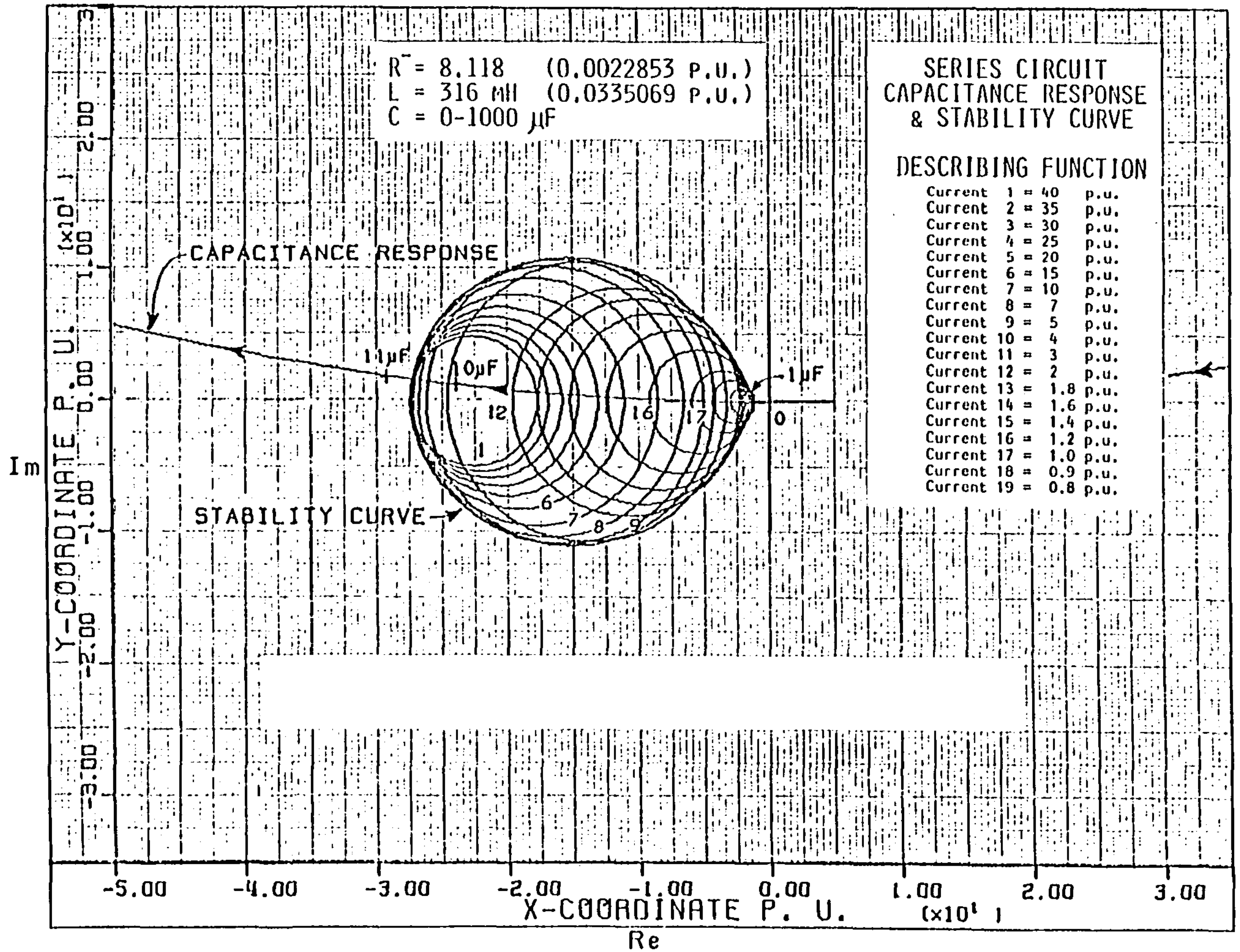


Figure 35. Sensitivity of capacitance response curve to decrease in resistance of the series circuit

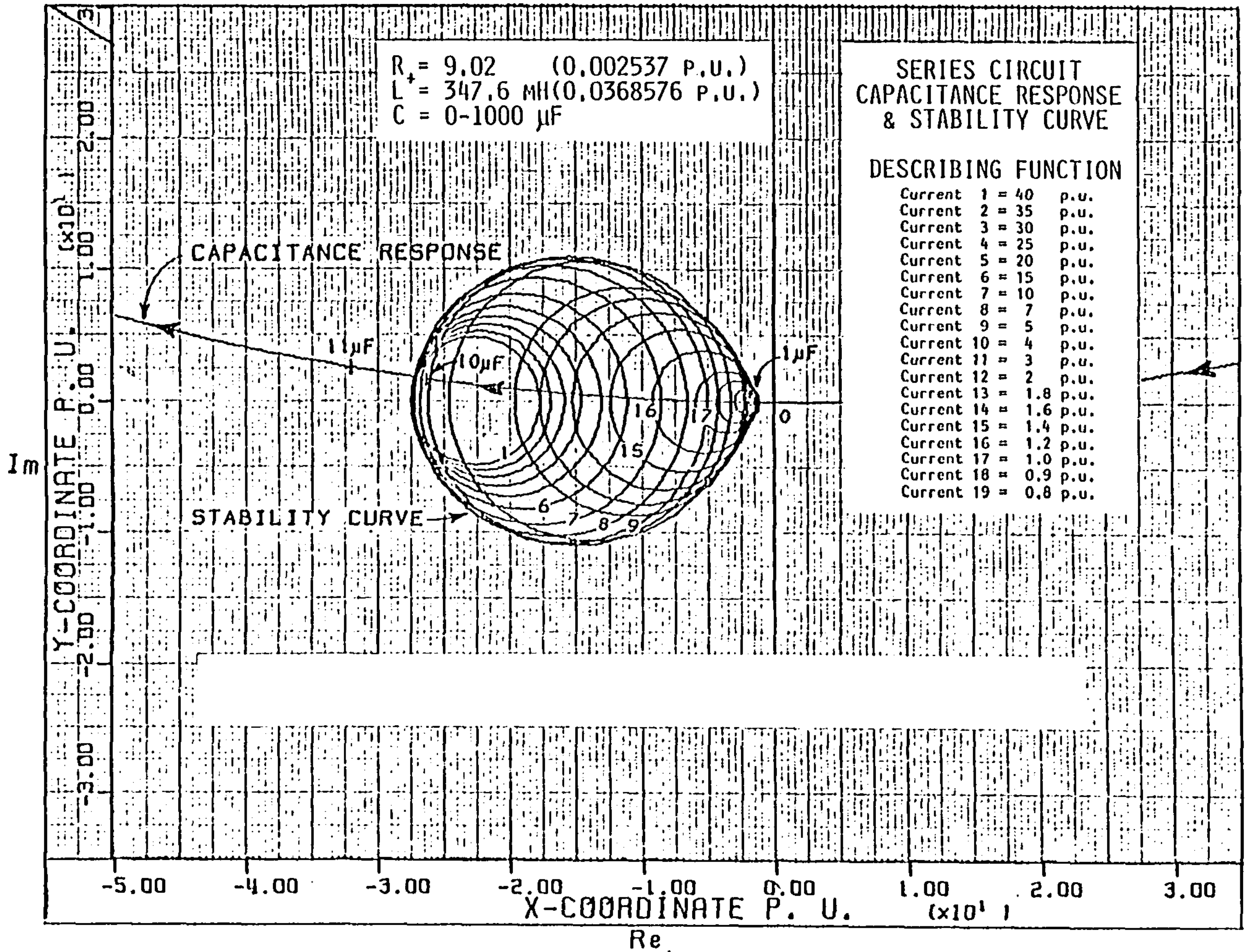
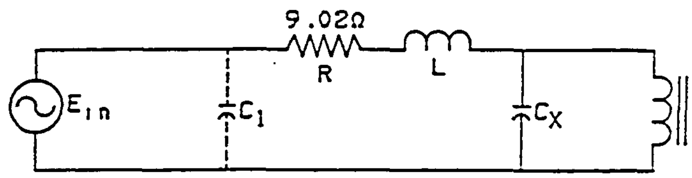


Figure 36. Sensitivity of capacitance response curve to increase in linear inductance of the series circuit

E. Determination of Critical Input Voltage

Applying phasor analysis to determine the critical supply voltage for which jump occurs in the given pi and series power circuits.

1. Pi circuit



$$E = (R+jX_L)I + \lambda_m W$$

$$= (R+jX_L)(I_T+I_C) + \lambda_m W$$

$$= (R+jX_L)\left(\frac{\lambda_m W}{jX_T} + \frac{\lambda_m W}{-jX_C}\right) + \lambda_m W$$

$$= (R+jWL)\left(\frac{\lambda_m W}{WL_T} - \frac{\lambda_m W}{j\frac{1}{WC}}\right) + \lambda_m W$$

$$= (R+jL)\left(\frac{\lambda_m}{jL} - \frac{\lambda_m}{j\frac{1}{C}}\right) + \lambda_m$$

$$E_C = (R+jL)\left(-j\frac{\lambda_{m_1}}{L_1} + jC\lambda_{m_1}\right) + \lambda_{m_1}$$

$$= -j\frac{\lambda_{m_1}}{L_1}(R+jL) + jC\lambda_{m_1}(R+jL) + \lambda_{m_1}$$

$$E_C = -j\frac{\lambda_{m_1}}{L_1}R + \frac{\lambda_{m_1}}{L_1}L + jC\lambda_{m_1}R - C\lambda_{m_1}L + \lambda_{m_1}$$

$$E_C = \lambda_{m_1}\left[\left(\frac{L}{L_1} + 1 - LC\right) + j\left(RC - \frac{R}{L_1}\right)\right] \quad (5.7)$$

$$E_C = 0.7650419\left[\frac{0.0335069}{0.3716672} + 1 - (0.0335069)(46.911916)\right.$$

$$\left. + j(0.002537)(46.911916) - \frac{0.002537}{0.3716672}\right]$$

$$E_C = -0.3685359 + j0.0858297$$

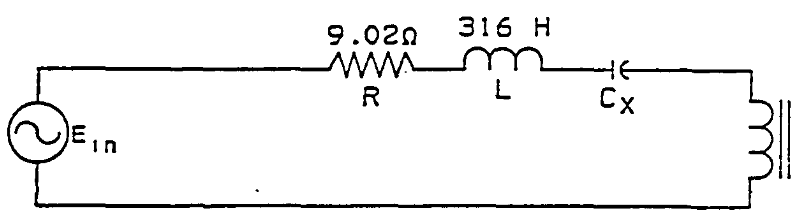
$$E_C = 0.378398 \ \angle 103°$$

$$E_C = E_B \cdot E_C \ (\text{P.U.})$$

$$E_C = (115)(0.378398)$$

$$E_C = 43.515828 \text{ V peak}$$

2. Series circuit



$$E_C = Im_1\left[R + jWL - j\frac{1}{WC} + jWL\right] \quad (5.8)$$

$$= 2.9002535[0.002537 + j0.0335069 - j\frac{1}{13.403404}$$

$$+ j0.309234]$$

$= 2.9002535[0.002537 + j0.0335069 - j0.0746073$

$+ j0.309234]$

$= 2.9002535[0.002537 + j0.2681337]$

$= 0.0073579 + j0.7776557$

$E_c = 0.7776905$ P.U.

$E_c = (0.7776905)(115)$

$= 89.434408$ V peak

F. Relative Severity of Jump Resonance

For the specified system inductance, resistance, and capacitance, the variation of lower and higher critical lambda or current jump points with the variation of capacitance of the system for a given line (constant inductance) is shown in Figures 37 and 38.

1. Pi circuit

It is clear that for higher capacitance value within its critical capacitance range, the relative jump severity and the critical supply voltage both increase.

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Jump to	Relative jump resonance	Pi ckt parameter
35 (0.3 P.U.)	78.5 (0.68 P.U.)	96.4 (0.84 P.U.)	23.5%	R = 9.02 Ω L = 316 mH C = 30 μF
48 (0.42 P.U.)	77.7 (0.67 P.U.)	109 (0.948 P.U.)	40.24%	R = 9.02 Ω L = 316 mH C = 35 μF
66 (0.574 P.U.)	80.5 (0.7 P.U.)	120 (1.04 P.U.)	49%	R = 9.02 Ω L = 316 mH C = 40 μF
130 (1.13 P.U.)	86.2 (0.75 P.U.)	134.5 (1.17 P.U.)	56%	R = 9.02 L = 316 mH C = 55 μF

Experiment No. 6:

Circuit's critical input voltage (VRMS	Transformer voltage (VRMS) Jump from	Jump to	Pi circuit parameter
35(0.3 P.U.)	78.5(0.68 P.U.)	96.4(0.84 P.U.)	R = 9.02 Ω L = 316 mH C = 30 μF

$$\% \text{ jump} = \frac{\lambda_2 - \lambda_1}{\lambda_1} \times 100$$

$$= \frac{0.84-0.68}{0.68} \times 100$$

$$= 23.5$$

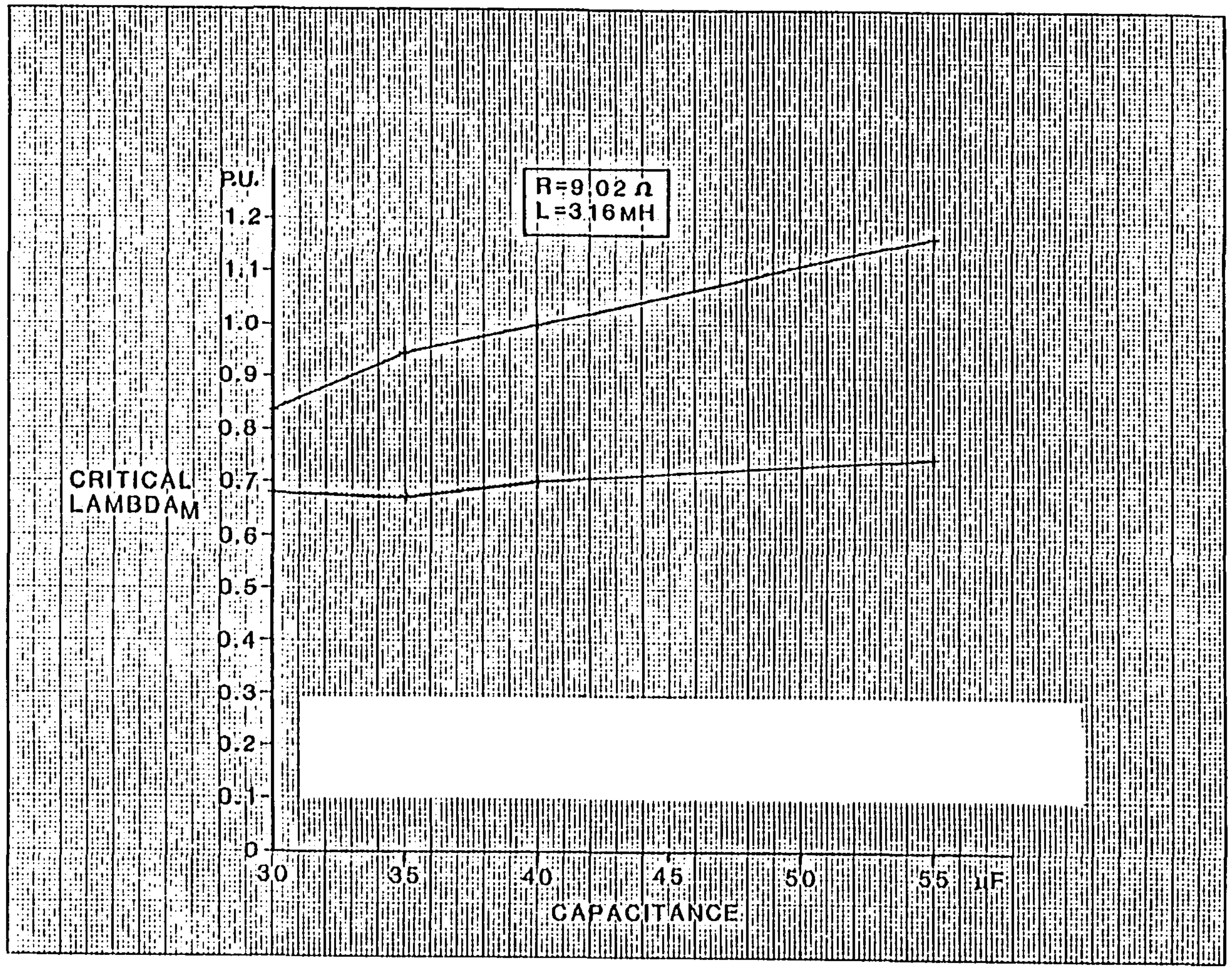


Figure 37. Locus of lower and higher hump lambdas as shunt capacitance varies for constant series inductance

Experiment No. 7:

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Jump to	Pi circuit parameter
48(0.42 P.U.)	77.7(0.67 P.U.)	109(0.948 P.U.)	R = 9.02 Ω L = 316 mH C = 35 μF

$$\% \text{ jump} = \frac{0.948-0.676}{0.676} \times 100$$

$$= 40.24$$

Experiment No. 8:

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Jump to	Pi circuit parameter
66(0.574 P.U.)	80.5(0.7 P.U.)	120(1.04 P.U.)	R = 9.02 Ω L = 316 mH C = 40 μF

$$\% \text{ jump} = \frac{1.04-0.7}{0.7} \times 100$$

$$= 49$$

Experiment No. 9:

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Jump to	Pi circuit parameter
130(1.13 P.U.)	86.2(0.75 P.U.)	134.5(1.17 P.U.)	R = 9.02 Ω L = 316 mH C = 55 μF

$$\% \text{ jump} = \frac{1.17-0.75}{0.75} \times 100$$

$$= 56$$

2. Series circuit

For the case of series circuit, as the capacitance value within the critical capacitance range increases, the critical supply voltage will also increase. However, the relative jump severity increases at first to a peak but drops as the critical capacitance value is further increased.

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Transformer voltage (VRMS) Jump to	Relative jump resonance	Series circuit parameter
45 (0.391 P.U.)	44.5 (0.387 P.U.)	61 (0.53 P.U.)	37%	R = 9.02 Ω L = 316 mH C = 5 μF
55 (0.478 P.U.)	80.5 (0.7 P.U.)	120 (1.043 P.U.)	49%	R = 9.02 Ω L = 316 mH C = 10 μF
95 (0.825 P.U.)	106 (0.922 P.U.)	147 (1.28 P.U.)	38%	R = 9.02 Ω L = 316 mH C = 15 μF

Experiment No. 2:

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Transformer voltage (VRMS) Jump to	Series circuit parameter
45(0.391 P.U.)	44.5(0.387 P.U.)	61(0.53 P.U.)	R = 9.02 Ω L = 316 mH C = 5 μF

$$\% \text{ voltage jump} = \frac{E_2 - E_1}{E_1} \times 100$$

$$= \frac{0.53 - 0.387}{0.387} \times 100$$

$$= 37$$

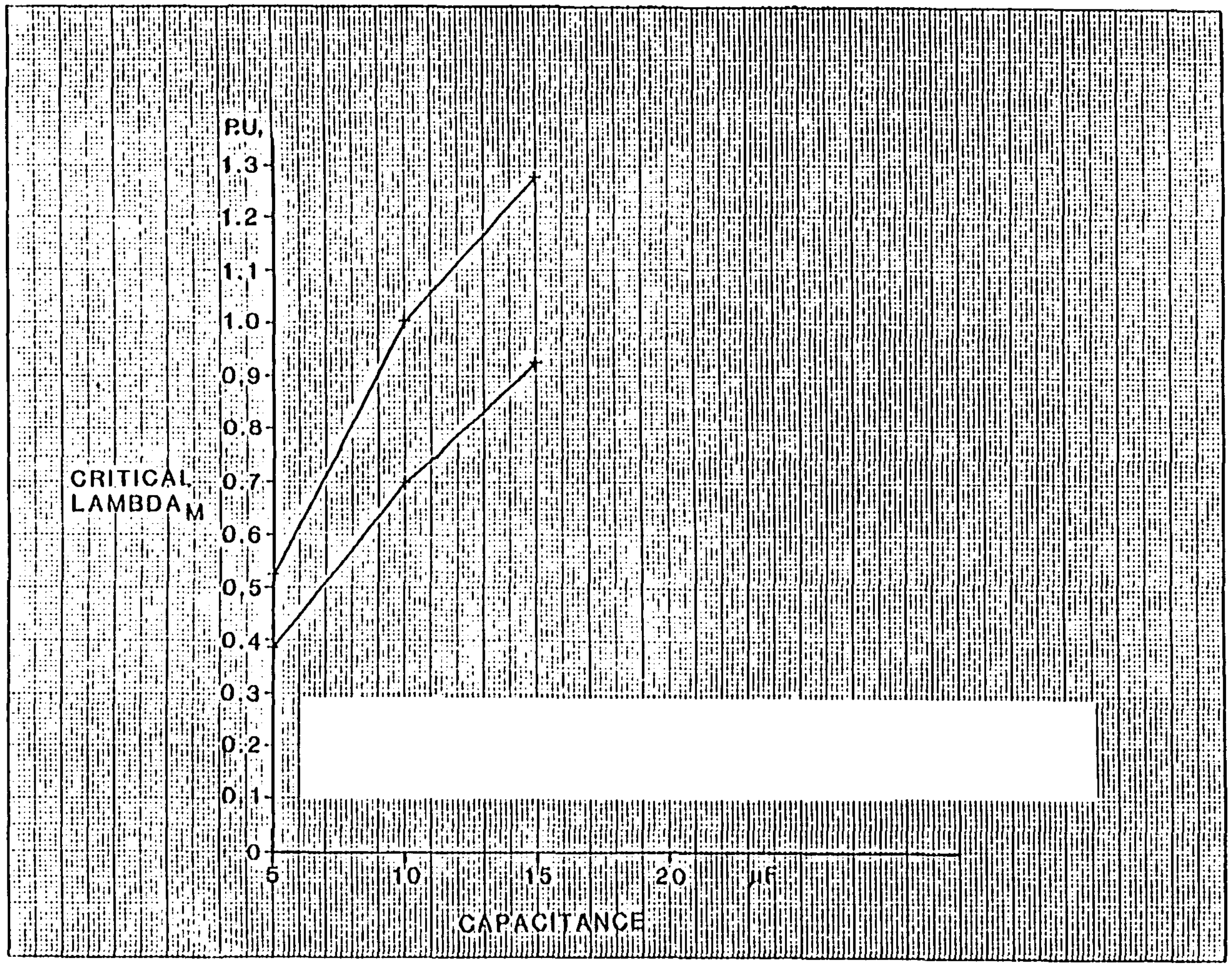


Figure 38. Locus of lower and higher jump lambda as series capacitance varies for constant series inductance

Experiment No. 6:

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Jump to	Series circuit parameter
55(0.478 P.U.)	80.5(0.7 P.U.)	120(1.043 P.U.)	R = 9.02 Ω L = 316 mH C = 10μF

$$\% \text{ voltage jump} = \frac{1.043-0.7}{0.7}$$

$$= 49.07$$

Experiment No. 7:

Circuit's critical input voltage (VRMS)	Transformer voltage (VRMS) Jump from	Jump to	Series circuit parameter
95(0.826 P.U.)	106(.922 P.U.)	147(1.28 P.U.)	R = 9.02 Ω L = 316 mH C = 15 μF

$$\% \text{ voltage jump} = \frac{1.28-0.922}{0.922} \times 100$$

$$= 38.64$$

G. Transformer Voltage Switching and Transient Times

In pi and series circuit experiments No. 7 and No. 6, the supply voltage was increased slowly until the critical supply voltage of 48 V and 55 V RMS was reached in about 70 s and 90 s, respectively. At that point, transformer voltage jumped from 77.7 V and 80.5 V to 109 V and 120 V RMS, respectively.

Swift's model and the present analysis were applied to find the switching time, i.e., the time required to reach the critical supply voltage, using same rate of supply voltage increase as that of the respective experiments as well as their calculated critical lambdas 0.765 P.U. and 0.9 P.U., respectively.

Also, for both experiments, transient times, i.e., time required to change from linear mode of critical λ_{m_1}, to that of the nonlinear mode of critical λ_{m_2}, are calculated for comparison with the experimental values observed on Figures 63, 64, 112, and 113.

As shown, there is a discrepancy between transformer's voltage transient time obtained by using G. W. Swift's model and that of the experiment.

1. Pi circuit

Method	Pi circuit	
	Switching	Transient
G. W. Swift's method	79.3 s	127.2 s
Present analysis	79.2 s	67.3 ms
Experimental	70 s	70 ms

In Experiment No. 7, as the variac output was increased to 48 VRMS in about 70 seconds, transformer voltage rose to 77.7 VRMS:

$$E_T = \frac{77.7}{115 \times 70 \times 377} t \text{ P.U.} \tag{5.9}$$

$$E_T = \frac{0.0096522}{377} t \text{ P.U.}$$

$$\lambda_m(t) = \frac{0.0096522}{377} t \text{ P.U.}$$

Since jump occurs at λ_{m_1} where

$$\lambda_{m_1} = 0.765 \text{ P.U.}$$

$$\therefore 0.765 = \frac{0.0096522}{377} t_1$$

$$t_1 = \frac{0.765 \times 377}{0.0096522} \text{ P.U.}$$

$$t_1 = \frac{0.765}{0.0096522} \text{ s}$$

$$t_1 = 79.26 \text{ s}$$

$$\therefore E_T = \frac{77.7}{115 \times 70 \times 377} t \qquad 0 < t \leq 79.26 \text{ s}$$

Time constant of the pi circuit is τ, where

$$\tau = \frac{2L}{R} \quad \text{Appendix E}$$

$$= \frac{2 \times 0.316}{9.02}$$

$$= 0.070 \text{ s}$$

since

$$V_{L_1} = 0.765 \text{ P.U.}$$

$$V_{L_2} = 2 \text{ P.U.}$$

$$\therefore\ 2 = 0.765\ e^{\frac{t}{\tau}}$$

$$\ln \frac{2}{0.765} = \frac{t}{\tau}$$

$$0.961 = \frac{t}{\tau}$$

$$t = 0.07 \times 0.961$$

$$= 0.0673 \text{ s}$$

$$= 67.3 \text{ ms}$$

This transient time is very close to the experimentally determined value of about 70 ms. However, as shown below using G. W. Swift's magnetization curve representation, the transient time of 127 s is too long compared with the experimental results.

$$\lambda_m(t) = \frac{77.7}{115 \times 70 \times 377}\ t \text{ P.U.} \qquad 0<t<79.26$$

$$\lambda_m(t) + 4\lambda_m(t)^5 = \frac{77.7}{115 \times 70 \times 377}\ t$$

$$+ 4\left(\frac{77.7}{115 \times 70 \times 377}\right)^5 t^5 \text{ P.U.} \quad t > 79.26 \text{ s}$$

Since jump occurs at λ_{m_1}, where

$$\lambda_{m_1} = 0.765 \text{ P.U.}$$

$$0.765 + 4(0.765)^5 = \frac{77.7}{115 \times 70 \times 377}\ t_1$$

$$+ 4\left(\frac{77.7}{115 \times 70 \times 377}\right)^5 t_1^5 \quad t \geq 79.26 \text{ s}$$

$$1.813 = 2.56 \times 10^{-5} t_1 + 44\text{x}10^{-23} t_1^5 = 0$$

or

$$4.4\text{x}10^{-23} t_1^5 + 2.56\text{x}10^{-5} t_1 - 1.813 = 0$$

$$t_1 = 29896.1 \text{ P.U.}$$

$$t_1 = 79.3 \text{ s}$$

This is in agreement with our calculated value, however, the transient time is quite different from the experimentally obtained value of ∿70 ms as shown:

$$\lambda_{m_2} = 2 \text{ P.U.}$$

$$2+4(2)^5 = \frac{77.7}{115 \times 70 \times 377} t_2 + 4\left(\frac{77.7}{115 \times 70 \times 377}\right)^5 t_2^5 \tag{5.10}$$

$$130 = 2.56\text{x}10^{-5} t_2 + 4.4\text{x}10^{-23} t_2^5$$

$$4.4\text{x}10^{-23} t_2^5 + 2.56\text{x}10^{-5} t_2 - 130 = 0$$

$$t_2 = 78121.94 \text{ P.U.}$$

$$t_2 = 207.22 \text{ s}$$

and

$$t_2 - t_1 = 127.22 \text{ s}$$

2. Series circuit

Similarly, the discrepancy of transformer voltage transient time obtained by using G. W. Swift's model from that of the experiment is shown:

Method	Series circuit	
	Switching	Transient
G. W. Swift's method	115.3 s	627.2 s
Present analysis	115.3 s	130.4 ms
Experimental	90 s	150 ms

In Experiment No. 6, as the variac output was increased to 55 VRMS in about 90 seconds, transformer voltage rose to 90.5 VMRS.

$$E_T = \frac{80.5}{115x90x377} \text{ t P.U.} \quad (5.11)$$

$$\lambda_m(t) = \frac{0.0077778}{377} \text{ t P.U.}$$

Since jump occurred at λ_{m_1} where

$$\lambda_{m_1} = I_{m_1} L_1$$

$$= 2.9002535 \text{ x } 0.3092347 \text{ P.U.}$$

$$\lambda_{m_1} = 0.896859 \text{ P.U.}$$

$$\therefore 0.896859 = \frac{0.0077778}{377} \text{ t}$$

$$t = \frac{377x0.896859}{0.0077778} \text{ P.U.}$$

$$= \frac{0.896859}{0.0077778} \text{ seconds}$$

$t = 115.3$ s

$$\therefore E_T = \frac{80.5}{115x90x377}\, t \qquad t \leq 115.3$$

circuit's time constant ≈ 0.070 s

= 70 ms

$$V_{L_1} = I_{m_1} L_1$$
$$= 2.9002535 \times 0.3092347$$
$$= 0.896859 \text{ P.U.}$$

$$V_{L_2} = I_{m_2} L_2$$
$$= 136.64002 \times 0.0422639$$
$$= 5.7749401 \text{ P.U.}$$

$$5.7749401 = 0.896859\ e^{\frac{t}{\tau}}$$

$$\ln \frac{5.7749401}{0.896859} = \frac{t}{\tau}$$

$$\frac{t}{\tau} = 1.8623845$$

$$t = 0.070 \times 1.8623845$$
$$= 0.1303669$$
$$\simeq 0.1304 \text{ s}$$
$$t = 130.4 \text{ ms}$$

It appeared that this transient time is closer to the experimentally determined value of 150 ms than 627s obtained by using G. W. Swift's magnetic curve representation as shown below:

$$\lambda_m(t\ ) = \frac{80.5}{115x90x377}\ t \quad P.U.$$

$$\lambda_m(t) + 4\lambda_m^5(t) = \frac{80.5}{115x90x377}t + 4(\frac{80.5}{115\ x\ 90\ x\ 377})t^5 \quad t>90s$$

Since jump occurred at λ_{m_1} where $\lambda_{m_1} = 0.896859$.

$$\therefore 0.896859 + 4(0.896859)^5 = \frac{80.5}{115x90x377}t_1$$

$$+ 4(\frac{80.5}{115x90x377})^5 t_1^5$$

$$3.2178895 = 2.06\ x\ 10^{-5}t_1 + 1.495\ x\ 10^{-23}t_1^5$$

or

$$1.495\ x\ 10^{-23}t_1^5 + 2.06\ x\ 10^{-5}t_1 - 3.218 = 0$$

$$t_1 = 43471.87\ P.U.$$

$$t_1 = 115.31\ s$$

This is in agreement with our experimental results. However, the transient time shown below is much longer than 150 ms.

$$\lambda_{m_2} = I_{m_2}L_2$$

$$= 136.64002\ x\ 0.0422639$$

$$= 5.7749401\ P.U.$$

$$5.7749401 + 4(5.7749401)^5 = \frac{80.5}{115x90x377}t_2$$

$$+ 4(\frac{80.5}{115x90x377})^5 t_2^5$$

$$5.774901 + 25691.97 = 2.06 \times 10^{-5} t_2 + 1.495 \times 10^{-23} t_2^5 \quad (5.12)$$

or

$$1.495 \times 10^{-23} t_2^5 + 2.06 \times 10^{-5} t_2 - 25697.745 = 0$$

$$t_2 = 279918.28 \text{ P.U.}$$

$$t_2 = 742.4888 \text{ s}$$

and

$$t_2 - t_1 = 742.4888 - 115.31$$

$$= 627.1788 \text{ s}$$

H. Phasor Analysis

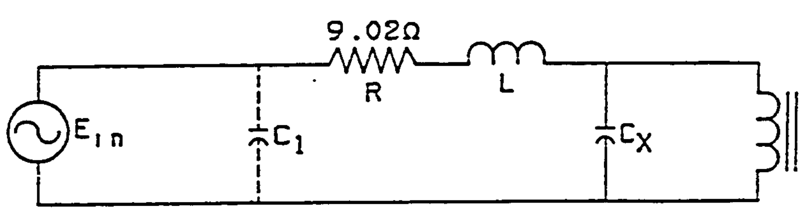
In what follows, the effect of ferroresonance on the equivalent impedance of the circuit is shown.

1. Pi circuit

In Experiment No. 7, for the critical parameters

$R = 9.02\ \Omega$,
$L = 316$ mH,
$C = 35\ \mu F$,
jump-from $\lambda_{m1} = 0.77$ P.U., and
$L_1 = 0.37$ P.U.

The corresponding Z_1 and Z_{in} are:



$$Z_1 = \frac{(-j75.786)(j1321.443)}{j1321.443-j75.786}$$

$$= -j80.399 \text{ capacitive}$$

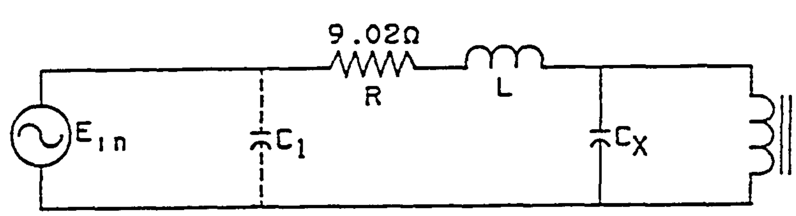
$$= -j80.4\ \Omega$$

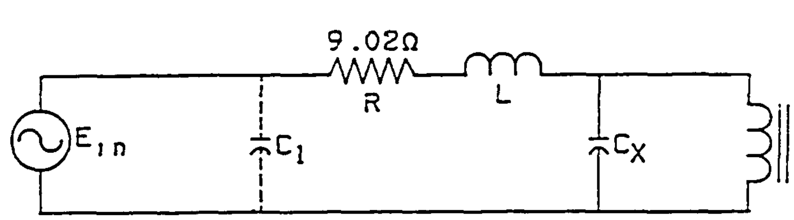
$$Z_{in} = 9.02 + j119.132 - j80.39$$

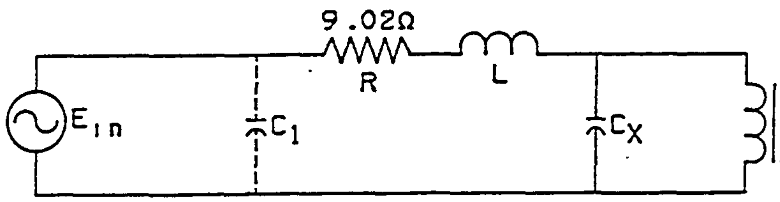
$$= 9.02 + j38.741925\ \Omega$$

Therefore, in this circuit, the current lags circuit's input voltage by an angle of 76.9°.

However, for jump-to λ_{m_2} = 2 P.U., L_2 = 0.0598 P.U., the corresponding Z_2 and Z_{in} are:







$$Z_2 = \frac{(-j75.786)(j212.6)}{-j75.786+j212.6}$$

$$Z_2 = -j117.7655$$

$$Z_{in} = 9.02 + j119.132 - j117.7665$$

$$Z_{in} = 9.02 + j1.365\ \Omega$$

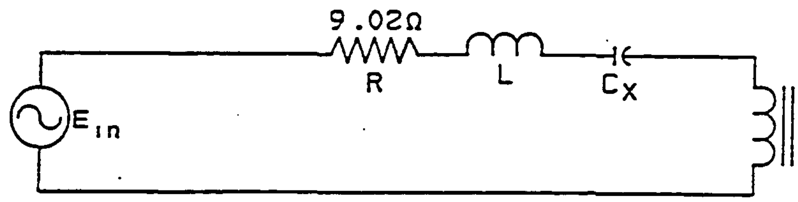
The circuit is still inductive and the current lags circuit's input voltage by an angle of 8.6°. However, it is possible with certain combinations of R, L, and C, that the circuit will be capacitive and the current then would be leading the voltage.

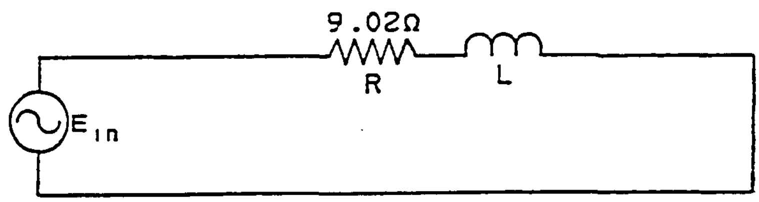
2. Series circuit

In Experiment No. 6, for the critical parameters,

$R = 9.02\ \Omega$,
$L = 316$ mH,
$C = 10\ \mu F$,
jump-from $I_{m_1} = 2.9$ P.U. and
$L_1 = 0.31$ P.U.

Therefore, Z_{in} is:

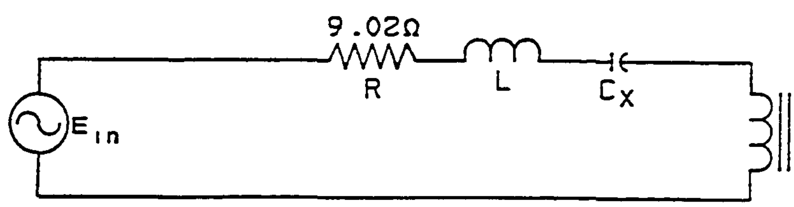


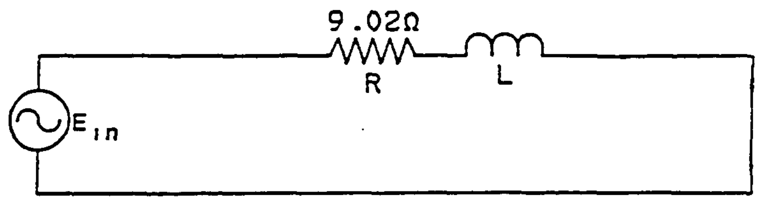


$$Z_{in} = 9.02 + j953.35$$
$$= 953.4\,\Omega \angle 89.5$$

Therefore, in this circuit, the current lags behind the input voltage by almost 90°.

However, for jump-to I_{m_2} = 136.6 P.U., L_2 = 0.0423 P.U., Z_{in} is:





$Z_{in} = 9.02 + j4.14707$

$= 9.9276673 \angle 24.69$

$Z_{in} = 9.93\ \Omega \angle 24.7$

$= 0.0027922$ P.U. $\angle 24.7$

The circuit is still inductive, and now the current lags the voltage by an angle of 24.7°. However, it is possible by choosing a combination of critical R, L, and C, such that this circuit will become capacitive and then the current will be leading the voltage.

I. Interpretation of Experimental Results

1. Pi circuit Experiment No. 7

For a 60 Hertz point at R = 9.02 Ω, L = 316 mH, and C = 35 μF,

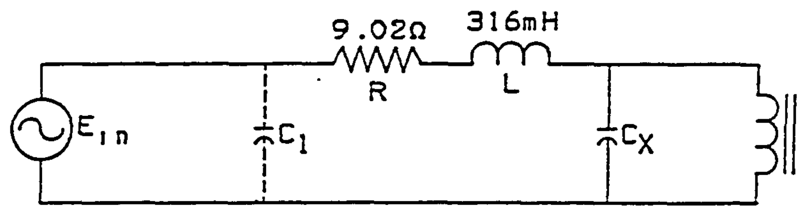
$L_1 = 0.3716672$ P.U.

$Z_1 = -j80.4\ \Omega$

and

$L_2 = 0.0597969$ P.U.

$Z_2 = -j117.76\ \Omega$



Up to inductance L_1, the impedance of the parallel section of the circuit is almost constant at about $Z_1 = -j80.4\ \Omega$. Therefore, increasing the voltage slowly will increase the current (I) slowly, as seen from the equation below:

$$I = \frac{V_0}{Z_1}$$

At the critical input voltage (E_c), the impedance of the parallel section of the circuit suddenly increases to $Z_2 = -117.76\ \Omega$. Since $I = I_c + I_T$ is constant at any time, then V_0 must increase suddenly to keep I constant in the equation:

$$\text{constant } I = \frac{V_{0\uparrow}}{Z_{2\uparrow}}$$

If V_0 increases suddenly, and the supply voltage is constant, $E-V_0$ decreases and therefore, I decreases suddenly.

After the jump, as the supply voltage is increased slowly, the current also decreases slowly.

2. Series circuit Experiment No. 6

For a 60 Hz point at R = 9.02 Ω, L = 316 mH, and C = 10 μF,

$$L_1 = 0.3092347 \text{ P.U.}$$

$$Z_1 = jWL_BL_1$$

$$Z_1 = j(377)\ (9.4308943)\ (0.3092347)$$

$$Z_1 = 1099.4676\ \Omega$$

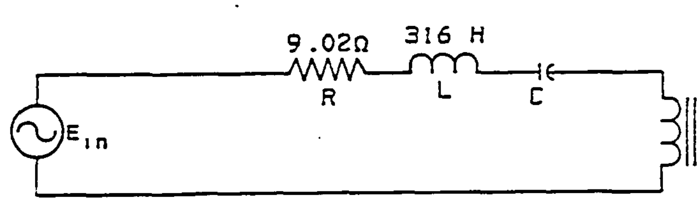
and

$$L_2 = 0.0422639 \text{ P.U.}$$

$$Z_2 = jWL_BL_2$$

$$Z_2 = j(377)\ (9.4308943)\ (0.0422639)$$

$$Z_2 = j150.26706\ \Omega$$



Up to inductance L_1, the impedance of the transformer and the circuit is almost constant at about Z_1 = 1099.5 Ω and 953.4 Ω, respectively. Therefore, increasing the voltage slowly will increase the current I, slowly as seen from the equation below:

$$I = \frac{E}{R + jWL - j\frac{1}{WC_x} + jWL_1} \qquad (5.13)$$

At the critical input voltage E_{C_1}, the impedance of the transformer and the circuit suddenly drops to 150.3 Ω and 9.93 Ω, respectively, and therefore, the current increases suddenly. Since

$$V_0 = (\overset{\uparrow}{I})(\overset{\downarrow}{WL_1})$$

Therefore, transformer voltage also increases suddenly.

After the jump, as the supply voltage is increased slowly, the current will also increase slowly.

J. Analytical vs. Experimental Prediction of Ferroresonance

As shown in the following data, our analysis indicates a close agreement between the analytical and the experimental results of predicting ferroresonance in the given circuit configurations:

1. Pi circuit

Analytical

Pi circuit parameters			Ferroresonance	
R (Ω)	L (mH)	C (μF)	Occurs	Does not occur
9.02	316	20		X
9.02	316	23.5	X	
9.02	316	35	X	
9.02	316	42	X	
9.02	316	45		X

Experimental

R (Ω)	L (mH)	C (μF)	Occurs	Does not occur
9.02	316	5		X
9.02	316	10		X
9.02	316	15		X
9.02	316	20		X
9.02	316	25		X
9.02	316	30	X	
9.02	316	35	X	
9.02	316	40	X	
9.02	316	45	X	
9.02	316	55	X	
9.02	316	65		X

2. Series circuit

Analytical

R (Ω)	L (mH)	C (μF)	Occurs	Does not occur
9.02	316	1	X	
9.02	316	10.6	X	
9.02	316	15		X

Experimental

R (Ω)	L (mH)	C (μF)	Occurs	Does not occur
9.02	316	5	X	
9.02	316	10	X	
9.02	316	15	X	
9.02	316	18		X

In summary:

Circuit	Lower limit Capacitance threshold (μF) Analytical	Lower limit Capacitance threshold (μF) Experimental	Upper limit Capacitance threshold (μF) Analytical	Upper limit Capacitance threshold (μF) Experimental
Pi	23.5	30	42	55
Series	1	5	10.6	15

K. Experimental Results

1. Critical Supply Voltage Level

It is evident from the data below that ferroresonance will occur at higher supply voltages for higher capacitance values within the critical capacitance range.

Circuit	Experiment No.	Critical input voltage (VRMS)	Critical capacitance value (μF)
Pi	7	48.5	35
	8	66	40
	9	130	55
Series	2	45	5
	6	58	10
	7	95	15

2. Effect of pi-circuit input capacitance on ferroresonance

To examine the effect of the pi-circuit input capacitance on ferroresonance, four experiments were conducted as follows:

Pi circuit parameters

$R = 9.02\ \Omega$ $C_2 = 10\ \mu F$
$L = 316$ mH
$C = 35\ \mu F$

$R = 9.02\ \Omega$ $C_2 = 30\ \mu F$
$L = 316$ mH
$C = 35\ \mu F$

$R = 9.02\ \Omega$ $C_2 = 10\ \mu F$
$L = 315$ mH
$C = 55\ \mu F$

$R = 9.02\ \Omega$ $C_2 = 50\ \mu F$
$L = 316$ mH
$C = 55\ \mu F$

It appears that input capacitance does not change the magnitudes of the transformer jump voltages.

3. Impact of Transformer Loading on ferroresonance

It appears from the following table that ferroresonance is also possible on lightly-loaded systems. Critical loads that could eliminate it must also be determined. However, it was observed that the nonlinear mode returned as soon as the resistance was disconnected.

a. Pi circuit

60 Hz point	Transformer load required
$R = 9.02\ \Omega$ $L = 316$ mH $C = 35\ \mu F$ $(X_C = 75.8\ \Omega = 0.0213$ P.U.	$R_L \leq 1965\ \Omega$ (0.553 P.U.)

R = 9.02 Ω
L = 316 mH
C = 40 μF (X_C = 66.3 Ω = 0.0186 P.U.) $R_L \le$ 710 Ω (0.2 P.U.)

R = 9.02 Ω
L = 316 mH
C = 55 μF (X_C = 48.2 Ω = 0.0136 P.U.) $R_L \le$ 225 Ω (0.063 P.U.)

For the 60 Hz point at R = 9.02 Ω, L = 316 mH, and C = 35 μF, connecting 1965 Ω resistor across the transformer will add a resistance to the impedance across the transformer as shown below:

$$Z_2 = \frac{(-j117.8)(1965)}{(1965-j117.8)}$$

$$= \frac{(-j117.8)(1965+j117.8)}{(1965-j117.8)(1965+j117.8)}$$

$$= \frac{-117.8(1965)^2 + (117.8)^2(1965)}{(1965)^2 + (117.8)^2}$$

$$= \frac{-j4.5485231 \times 10^8 + 27277991}{(1965)^2 + (117.8)^2}$$

$$= \frac{27267991 - j4.5485231 \times 10^8}{3875101.8}$$

$$Z_2 = 7.036759 - j117.37816$$

The capacitive reactance is almost the same, except that the effective resistance of the circuit has been increased to 16.056716 Ω. The new resistance value is no longer a critical value.

To eliminate ferroresonance, any resistance connected across the transformer should have values on or below the lines shown on Figure 39.

b. Series circuit Ferroresonance was also observed on the series circuit when transformer was loaded with greater than 690 Ω resistance. However, when 690 Ω or smaller resistance was connected across the transformer it completely eliminated the nonlinear mode.

60 Hz point	Transformer load required
$R = 9.02\ \Omega$ $L = 316$ mH $C = 15\ \mu F$	$R_L \leq 690\ \Omega$

For the 60 Hz point, 9.02 Ω, L = 316 mH, and C = 15 μF, connecting a 690 Ω resistor across the transformer will add a resistance to the impedance, but it will preserve the original reactance value:

$$Z_2 = \frac{(j150.3)(690)}{690+j150.3}$$

$$= \frac{(j150.3)(690)(690-j150.3)}{(690+j150.3)(690-j150.3)}$$

$$= \frac{j(690)^2(150.3)+(150.3)^2(690)}{690^2+(150.3)^2}$$

$$= \frac{j71557830+15587162}{476100+22590.09}$$

$$= \frac{15587162+j71557830}{498690.09}$$

$$= 31.25621 + j143.49150$$

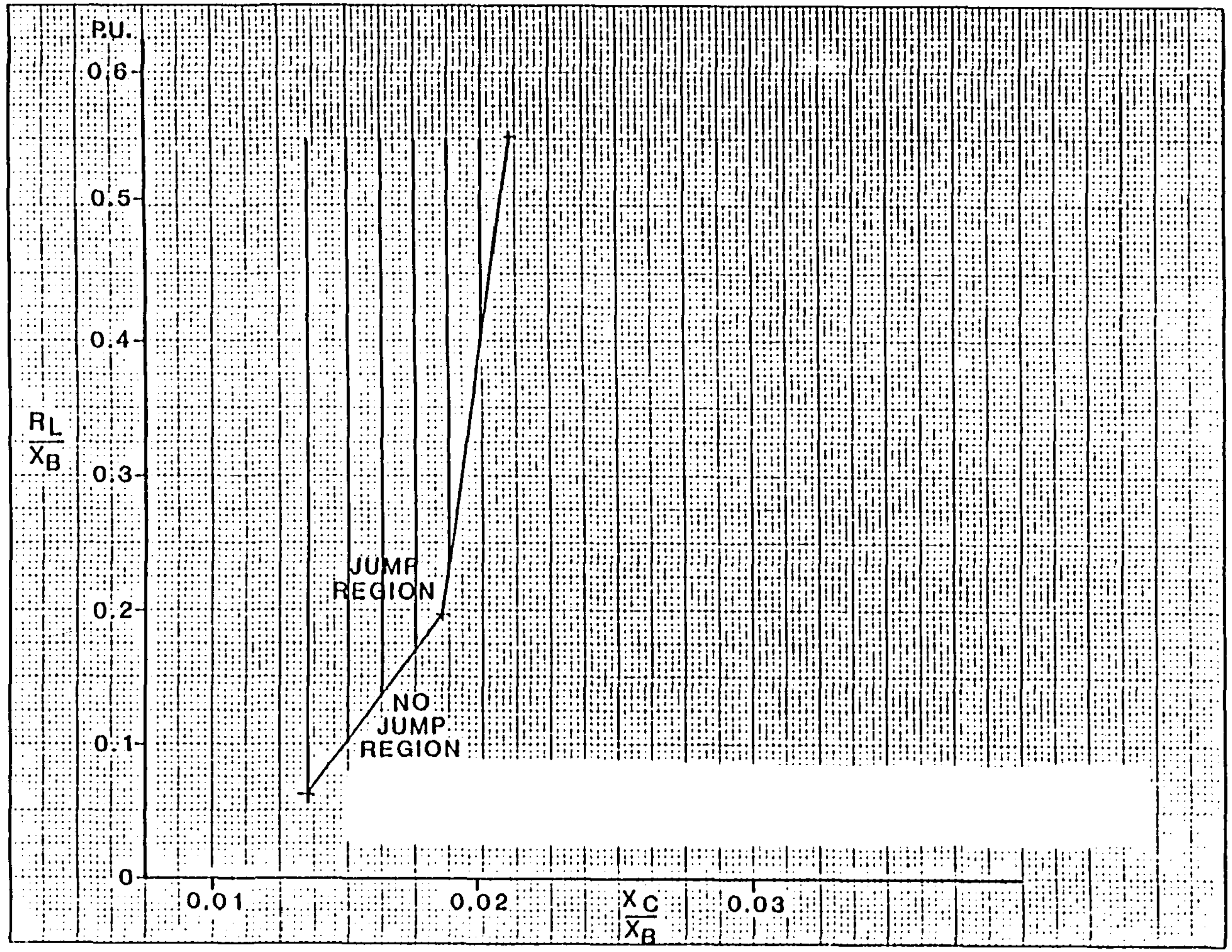


Figure 39. Load resistance to control ferroresonant over-voltage in the pi circuit

The reactance is almost the same like before, except the effective resistance of the circuit has been increased to R_T = 9.02 + 31.2562, and this value of resistance is no longer a critical value.

4. Switching modes by a circuit breaker

a. Pi circuit The following experiments were conducted under two sets of linear parameters, (1) R = 9.02 Ω, L = 316 mH, and C = 35 μF; (2) R = 9.02 Ω, L = 316 mH, and C = 55 μF.

At certain variac output voltages, modes were switched by turning the circuit breaker on and off.

In these experiments, the first interesting observation as the variac output was approaching the critical input voltage, the switching ease from one mode to another was completely reversed. At well below the critical input voltage and with more than 100 breaker switchings, it was not possible to switch transformer voltage from linear to nonlinear mode. However, as variac output increased but remained below the critical input voltage (48 VRMS for the first critical set, 130 VRMS for the second set) it became more difficult to switch from the linear to the nonlinear mode than the reverse. However, as the variac output increased still further toward the input critical voltage, the tendency to swtich back and forth almost equalized until at just before the critical input voltage it becomes more difficult to switch from a nonlinear to a linear mode. At critical input voltage and with more than 100 breaker switchings, it was not possible to switch back transformer voltage from nonlinear to linear mode, i.e., the switching tendency has been completely reversed.

The second observation is that it might seem that switching should not even occur at or above 55 μF because these are not critical values. In fact, during breaker on and off, transient charging currents are created, which changes the current through the nonlinear element. Therefore, the inductance which is a function of the current will also change. This violates our original assumption that these values were determined under steady-state conditions.

The third observation was that the critical linear parameters determined under steady-state were also valid for the transient condition.

In summary:

Pi circuit	R = 9.02 Ω L = 316 mH C = 35 μF	Linear mode LM Nonlinear mode NLM	R = 9.02 Ω L = 316 mH C = 55 μF

<table>
<tr><th>Circuit's critical input voltage (VMRS)</th><th>Mode switching</th><th>Circuit's critical input voltage (VMRS)</th><th>Mode switching</th></tr>
<tr><td>0-40</td><td>LM-N→NLM</td><td>0-60</td><td>LM-N→NLM</td></tr>
<tr><td colspan="4">For supply voltages of 0-40 or 0-60, voltage jumps from linear to nonlinear mode did not occur when breaker was switched on and off for more than 300 times.</td></tr>
<tr><td>42</td><td>$LM \overset{D}{\rightleftarrows} NLM$</td><td>70</td><td>$LM \overset{D}{\rightleftarrows} NLM$</td></tr>
<tr><td colspan="4">At 42 V and 70 V, it took less than 300 switchings of the breaker for the voltage to jump from linear to nonlinear mode, however, it took fewer times for the voltage to jump back to linear mode.</td></tr>
<tr><td>44</td><td>$LM \rightleftarrows NLM$</td><td>80</td><td>$LM \rightleftarrows NLM$</td></tr>
<tr><td colspan="4">At 44 V and 80 V, it almost took the same number of switchings of the breaker to change the modes either way.</td></tr>
<tr><td>46-47</td><td>$LM \underset{D}{\rightleftarrows} NLM$</td><td>90-120</td><td>$LM \underset{D}{\rightleftarrows} NLM$</td></tr>
<tr><td colspan="4">At 46-47 V or 90-120 V, the switching tendency is completely reverse of that at 42 V or 70 V.</td></tr>
<tr><td>48(CIV)-130</td><td>LM←N-NLM</td><td>130(CIV)</td><td>LM←N-NLM</td></tr>
<tr><td colspan="4">At 48 V and 130 V or greater, the switching tendency is completely reverse of that at 0-40 V or 0-60 V.</td></tr>
</table>

<u>b. Series circuit</u> To find out the relative ease at which a mode would switch from linear to nonlinear or vice versa, below, at, or above the critical input voltage, the following experiments were conducted under three sets of linear parameters.

(1) R = 9.02 Ω, L = 316 mH, and C = 3.3 μF

(2) R = 9.02 Ω, L = 316 mH, and C = 15 μF, and

(3) R = 9.02 Ω, L = 316 mH, and C = 18 μF.

For the first and third sets of parameters, it was not possible to switch from the linear to nonlinear mode with over a 100 times turning the breaker on and off. However, for the second set at below critical input voltage (95 volts RMS for R = 9.02 Ω, L = 316 mH, and C = 15 μF), it was again difficult to switch from the linear to nonlinear mode. As the input voltage was increased to the critical input voltage of 96 Volts RMS, the ease of switching from linear to nonlinear or vice versa was the same. However, at above critical input voltage (97 Volts RMS) and with more than 100 transformer switchings it was not possible to switch back transformer voltage from nonlinear to linear mode.

Circuit's critical input voltage (VMRS)	Mode switching	Circuit's critical input voltage (VMRS)	Mode switching	Circuit's critical input voltage (VMRS)	Mode switching
0-135	LM-N→NLM	0-94	LM-N→NLM	0-135	LM

Supply voltages of 0 to 135 V in the first and third experiments did not cause voltage jump when the breaker was switched on and off for more than 300 times.

However, for the second experiment, this was true only for supply voltage of 0 to 94 V.

95 LM $\overset{D}{\rightleftarrows}$ NLM

At 95 V, it took less than 300 switchings of the breaker, for the voltage to jump from linear to nonlinear mode; however, it took fewer times for the voltage to jump back to linear mode.

95(CIV)97 LM←N-NLM

At 95 V to 97 V, the switching tendency is completely reverse of that at 94 V. Therefore, if ferroresonance is suspected in a given circuit, then switching in that circuit becomes of prime importance. To avoid ferroresonance in this case, switching should be either minimized or condition of the supply changed to a safe switching.

L. Experimental vs. Methods Results

1. Jump resonance and the critical input voltage of the circuit

Summary of the results obtained by various methods based on a common set of critical parameters and their magnetization curve representation:

Common pi circuit parameters

$R = 0.002$ P.U.
$L = 0.025$ P.U.
$C = 50$ P.U.

Method	Magnetization curve representation
Piecewise linearization technique	$m_1 = 1$ P.U., $m_2 = 27$ P.U.
G. W. Swift's method	$I_L = \lambda + 4\lambda^5$
Present analysis	$N(\lambda_m)$

Method	Transformer voltage (P.U.) Jump from	Jump to	Circuit's critical input voltage (P.U.)
G. W. Swift's method	0.96	1.38	0.19
Piecewise linear method	0.89	1.595	0.22
Present analysis	0.94	1.46	0.22
Experiment	1.08	1.44	0.17

2. Transformer voltage switching and transient times

The results of G. W. Swift's method, present analysis and the experiment are shown:

Method	Pi circuit Switching	Pi circuit Transient	Series circuit Switching	Series circuit Transient
G. W. Swift's method	79.3 s	127.2 s	115.3 s	627.2 s
Present analysis	79.26 s	67.3 ms	115.3 s	130.4 ms
Experimental	70 s	70 ms	90 s	150 ms

It appears that the transient time obtained by using Swift's method disagrees with the experimentally observed transient time.

3. Jump resonance, critical supply voltage, and voltage transient time of the transformer

The results of Gear's digital program, present analysis, and the experiment are based on a common set of critical parameters and the magnetization curve representation.

Common pi circuit's critical parameters

$R = 9.02\Omega (0.002537 \text{ P.U.})$
$L = 316 \text{ mH } (0.0335069 \text{ P.U.})$
$C = 35\ \mu F\ (46.911916 \text{ P.U.})$

Method	Magnetization curve representation
Gear's method	$I_L = \lambda + 9\lambda^3 + 4\lambda^5$
Present analysis	$N(\lambda_m)$

Method	Transformer peak voltage (P.U.) Jump from	Jump to	Circuit's critical input peak voltage (P.U.)	Transformer voltage transient time
Gear's method	0.48	1.16	0.21	567 ms
Present analysis	0.8	1.7	0.38	67.3 ms
Experimental	0.96	1.34	0.59	70 ms

VI. CONCLUSIONS AND RECOMMENDATIONS

There are several methods to study ferroresonance such as piecewise linearization techniques and the Incremental Describing Function of G. W. Swift (26). In these methods, the magnetization characteristic model appears to be important in the evaluation of the possible occurrence of ferroresonance.

In piecewise linearization and G. W. Swift's method, the ferroresonance problem of a pi or series unloaded circuit is approached by simplifying the circuit and the nonlinearity representation in order to make the mathematics reasonable.

Several researchers in the past have used two straight lines to represent nonlinearity. These studies were conducted between 1931 and 1978, and appears to be focused on series circuits consisting of a resistor, capacitor, and an iron-cored inductor. While power transformers are now connected to either a transmission or distribution line, implying the presence of an inductance in the transformer's circuit, these studies have excluded this inductance that influences the occurrence of ferroresonance. Also, these studies appear not to emphasize the linear parameters R, L, and C of the system which appears to be as important as the magnetization curve for the analysis of ferroresonance. In summary, the circuits used in past studies may not exactly represent a power circuit.

The critical lambdas obtained by this study and based on common parameters appear closer to the experimental values than those of the two straight line approximations of the magnetization curve. In general, as the number of linear segments for the magnetization curve representation increased it becomes more difficult to obtain a solution.

Objections to the graphical solution of critical lambda by G. W. Swift is that it may provide pessimistic results concerning the susceptibility of various circuits to ferroresonance. It would be unrealistic to represent different sizes of transformers made up of different core material composition with the same single quintic non-linear term.

Swift's representation of a magnetization curve by one nonlinear quintic term may limit the application to a specific type of transformer.

Gear's program, as a tool for the utilities to study ferroresonance, is inconvenient because in a limited time the solution may not be available.

However, if we represent the transformer by a Describing Function based on two slopes of the magnetization curve in a pi or series circuit, it would then be possible to obtain the critical lambdas or currents analytically.

Also, a good advantage is provided by this method because the solution procedure does not break down in the sense of becoming much more difficult for systems of higher order than the third.

The Incremental Input Describing Function of the two-slope representation of the magnetization curve although an approximation, is shown to yield reasonable results for the determination of the onset of ferroresonance instabilities in laboratory simulated power circuits.

This Describing Function of the transformer based on two slopes of the magnetization curve has been used to study fundamental ferroresonance in a given pi or series circuit under unloaded as well as loaded circuit conditions. Therefore, in a given circuit configuration with known parameters, it is possible to predict:

1. Whether or not ferroresonance will occur for the given R, L, and C circuit.

2. Range of critical capacitance values for a given R and L.

3. If ferroresonance occurs in a given circuit, the method shows how far the given parameters should be changed and in what direction in order to eliminate its occurrence.

4. Analytical and graphical values of critical λ_m, I_m, and L_{eq}.

5. Sensitivity of the 60 Hertz capacitance response locus, G(1j,R,L,C) to variation in R and L.

6. Critical input voltage above which ferroresonance will occur and transformer voltage just before and after ferroresonance.

In this study, the results obtained by (1) piecewise linearization technique, (2) G. W. Swift's method, (3) Gear's digital program, and (4) two-segments B-H curve methodology were compared with the experimental results described in this study to indicate the various preferable approaches to study ferroresonance. The following summarizes the various observations resulted from this study:

1. There is a discrepancy between transformer voltage transient time obtained by using G. W. Swift's method and that observed experimentally.

2. Magnitudes of critical λ_m obtained by direct two-slope piecewise linearization method are less accurate than those given by the alternate methodology when compared to those of the experiment.

3. The critical jump-to λ_m obtained by the Gear's digital program is close to that of the experiment, however, jump-from λ_m, critical input voltage, and transient time are not. These results could have been improved if the digital program utilized a polynomial more accurately matched to the core characteristic than the approximation $\lambda+9\lambda^3+4\lambda^5$.

4. Although methods 1, 2, and 3 provide close jump-to λ_m to those of the experiment, they

are limiting or limited in their application to ferroresonance studies of the power system.

5. In the pi circuit, there are two capacitance threshold values, the upper and lower limits, that will place the system in the ferroresonance region while there is only an upper limit in the series circuit.

6. For the pi circuit, increasing the circuit's resistance or decreasing its inductance, decreases the circuit's susceptibility to ferroresonance and vice versa. In the series circuit, this susceptibility is more sensitive to variation in inductance than the resistance.

7. The following critical parameter's value, validated by experiments, caused ferroresonance in the given configuration:

   a. Pi circuit

Parameter	Predicted	Experimental
L	316 mH	316 mH
R	9.02 Ω	9.02 Ω
C	23.5 to 42 μF	30 to 55 μF

   b. Series circuit

Parameter	Predicted	Experimental
L	316 mH	316 mH
R	9.02 Ω	9.02 Ω
C	1 to 10.6 μF	5 to 15 μF

8. For the specified parameters in the given pi circuit configuration, the relative severity of voltage jump resonance increased as the capacitance value within the critical capacitance range increased. However, in the series circuit, the relative severity of voltage jump increased to a peak and dropped for further increases in the capacitance value.

9. Minimum transformer loading is required to eliminate ferroresonance over voltage and the required resistance to eliminate ferroresonance depends on the critical capacitance value. As shown in Figure 39, these values must be on or below the lines on the graph.

10. The calculated transformer voltage switching and transient times for the pi and series circuit configurations are close to those of the experiments as shown:

Circuit configuration	Transformer voltage switching time Calc.	Transformer voltage switching time Exper.	Transformer voltage transient time Calc.	Transformer voltage transient time Exper.
pi	79.26s	70s	67.3ms	70 ms
series	115.31s	90s	130.4ms	150ms

11. There is a noticeable disagreement between the voltage transient time obtained by G. W. Swift's method and that of the experiment:

Circuit Configuration	Transformer voltage Calculated	Transformer voltage Experimental
pi	127.2s	70ms
series	627.2s	150ms

12. If ferroresonance is suspected in a given circuit, then switching in that circuit becomes of prime importance. To avoid ferroresonance, in this case, switching should be either minimized or condition of the circuit changed to a safe switching.

13. Ferroresonance will occur at higher supply voltages for higher capacitance values within the critical capacitance range.

14. Gear's program for the utilities to use as a tool to study ferroresonance is inconvenient because it depends on the process of trial and error that is time consuming.

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IX. APPENDIX A: EXPERIMENTS

A. Experimental Validation of the Critical Capacitance Range in Pi Circuit

To verify that 23.5-42 μF capacitance range of values will actually cause ferroresonance in the typical pi-circuit configuration, values for capacitance 5, 10, 15, 20, 25, 30, 35, 40, 55 and 65 μF were chosen for the following experiments with values of λ_0, m_1, and m_2 obtained from the B-H curve:

λ_0 = 0.6551724 P.U.

m_1 = 1 P.U.

m_2 = 27.454377 P.U.

and the calculated values used

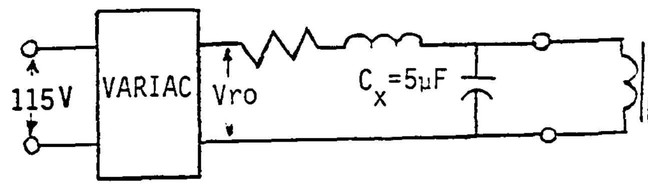
R = 9.02 Ω (0.002537 P.U.)

L = 316 mH (0.0335069 P.U.)

The only capacitance values that caused ferroresonance were between 30 to 55 μF.

Experiment No. 1:

R = 9.02 Ω
L = 316 mH
C_X= 5 μF



Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	8.7	6.3	0
20	13.8	8.6	0.03
25	18.8	10	0.04
30	24	11.7	0.05
35	28.8	13.8	0.07
40	34.3	15.7	0.095
45	39.3	17.4	0.105
50	42.5	18.5	0.11
55	47.3	20.3	0.115
60	52.3	22.2	0.12
65	56.8	24	0.135
70	61.2	26	
75	66	28.1	0.16
80	70	30	0.175
85	74.5	33	0.19
90	78.9	36	0.208
95	83.7	41	0.227
100	87.7	46.7	0.245
105	92.9	54.1	0.272
110	98.2	62.7	0.32
115	104.7	71.7	0.33
120	110.5	80.3	0.36
125	117.3	88.2	0.392
130	122	94	0.414
135	126	97.9	0.422
137	127	99	0.438

No Ferroresonance

Start experiment

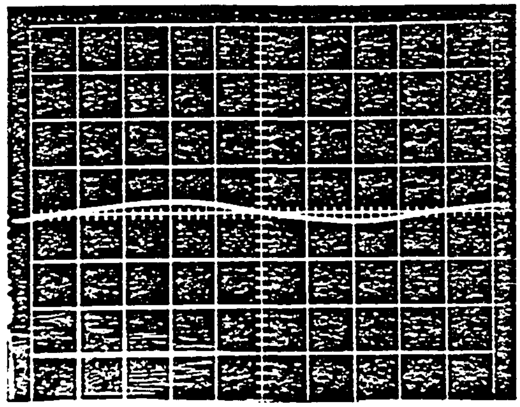


Figure 40. Voltage across transformer (V_T) variac output 15 VRMS, C = 0 μF (scale 50 V/Division)

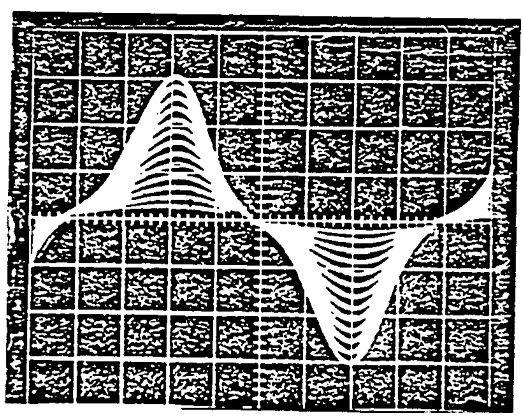


Figure 41. Voltage across transformer (V_T) variac output between 15 V - 135 VRMS, voltage increments of 10 VRMS (C = 0 μF) (scale = 50 V/Division)

Experiment No. 1

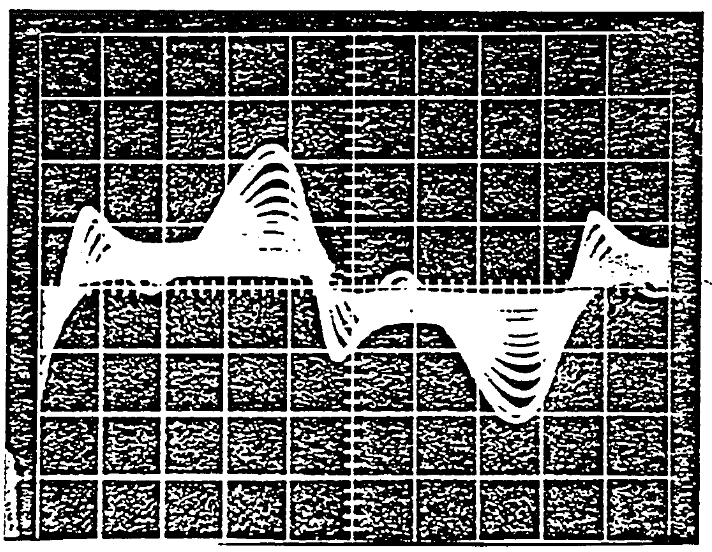


Figure 42. Voltage across transformer (V_T) variac output between 15 V - 135 VRMS (voltage increments of 10 VRMS) (C = 5 μF) (scale 100 V/Division)

Experiment No. 1

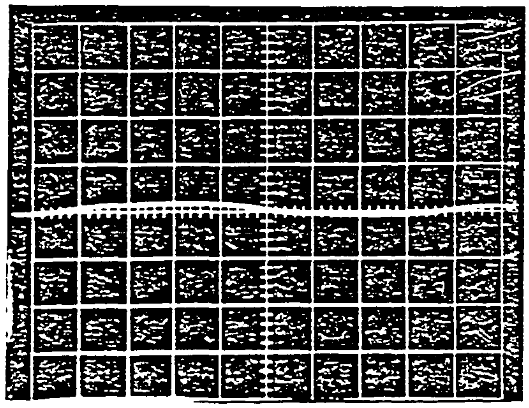


Figure 43. Voltage across transformer (V_T) variac output 15 VRMS (C = 5 μF) (scale 50 V/Division)

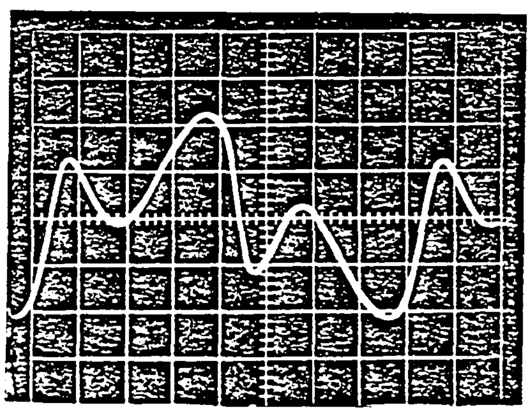


Figure 44. Voltage across transformer (V_T) variac output 135 VRMS (C = μF) (scale 100 V/Division)

Experiment No. 2:

$R = 9.02\ \Omega$
$L = 316$ mH
$C_X = 10\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	10	5.2	0
20	16.5	7	0.025
25	22.8	8.5	0.03
30	29.2	11.2	0.05
35	36	12.2	0.08
40	42.5	14	0.1
45	47.3	17	0.115
50	53.2	19	0.12
55	59	20.8	0.13
60	64.2	22.2	0.15
65	69.2	23.8	0.165
70	73.3	25	0.18
75	77.7	26.3	0.185
80	81	27.8	0.19
85	84	29.6	0.195
90	86.7	31.3	0.2
95	89	33.8	0.21
100	91.2	36.3	0.217
105	93.4	39.2	0.233
110	95.2	42	0.25
115	97.3	46	0.27
120	99.6	51	0.292
125	101	55	0.31
130	103	59.8	0.331
135	105.5	65.8	0.363
137	107.2	69.7	0.378

No Ferroresonance

Experiment No. 2

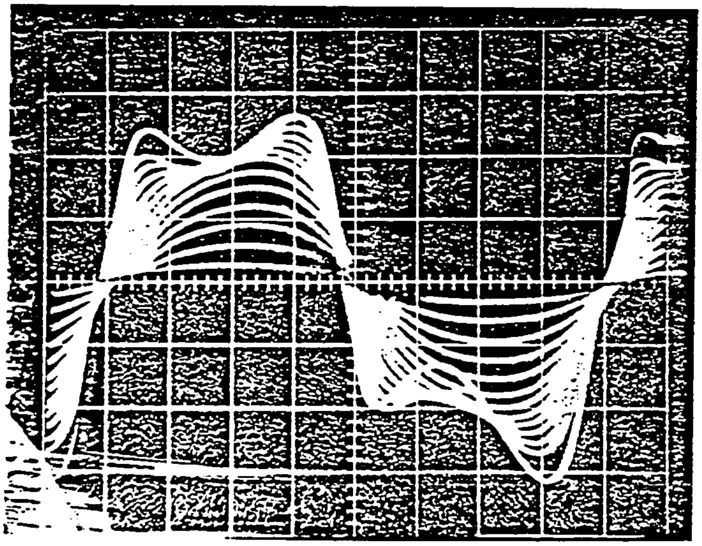


Figure 45. Voltage across transformer (V_T) (variac output between 15 V - 135 VRMS, voltage increments of 10 VRMS, voltage increments of 10 VRMS, C = 10 μF, scale = 50 V/Division

Experiment No. 2

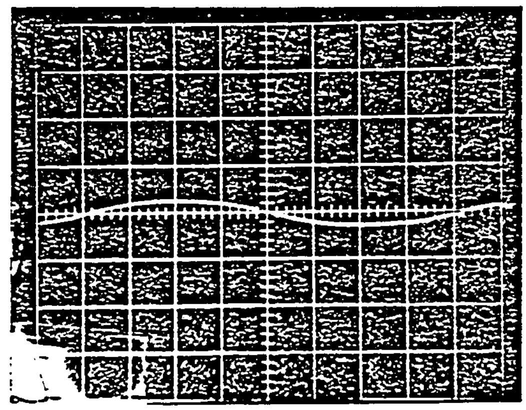


Figure 46. Voltage across transformer (V_T) (variac output 15 VRMS, C = 10 μF, scale = 50 V/Division)

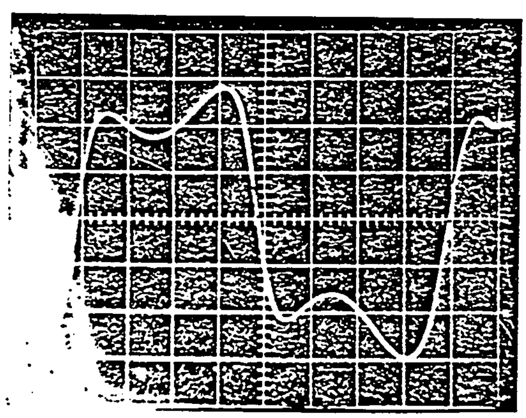


Figure 47. Voltage across transformer (V_T) (variac output 135 VRMS, C = 10 μF, scale 50 V/Division)

Experiment No. 3:

$R = 9.02\ \Omega$
$L = 316$ mH
$C_X = 15.06\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	12.7	5.7	0.0
20	22	10	0.05
25	30.2	14.7	0.1
30	39	19.9	0.13
35	45.5	24.5	0.17
40	54.6	29	0.2
45	61.6	32	0.229
50	69	34.4	0.248
55	75	36.2	0.263
60	80	36.3	0.265
65	83.7	36.5	0.266
70	86.7	34.3	0.26
75	89.2	33	0.257
80	91.5	32	0.248
85	93.5	32	0.24
90	95.2	36.2	0.235
95	97	33.3	0.231
100	98.7	35.3	0.23
105	100	37.1	0.236
110	101.8	39.5	0.292
115	103	42	0.253
120	104	45	0.267
125	105.5	48	0.285
130	106.8	51	0.31
135	108.0	54.5	0.32
138	108.8	56.8	0.333

No ferroresonance

Experiment No. 3

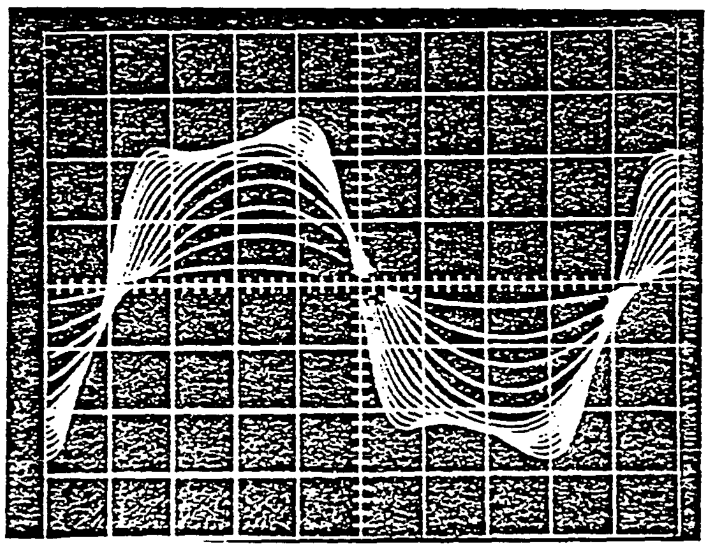


Figure 48. Voltage across transformer (V_T) (variac output between 15 V - 135 VRMS, voltage increments of 10 VRMS, C = 15 μF, scale 50 V/Division)

Experiment No. 3

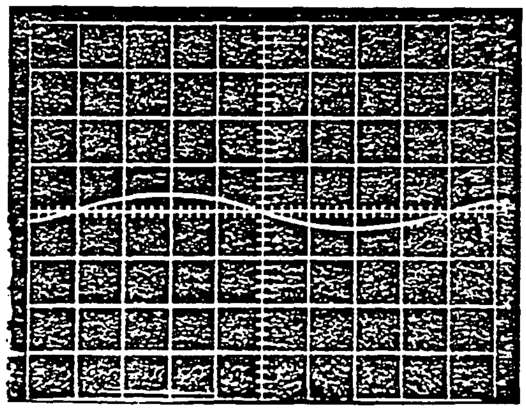


Figure 49. Voltage across transformer (V_T) (variac output 15 VRMS, C = 15 μF, scale = 50 V/Division

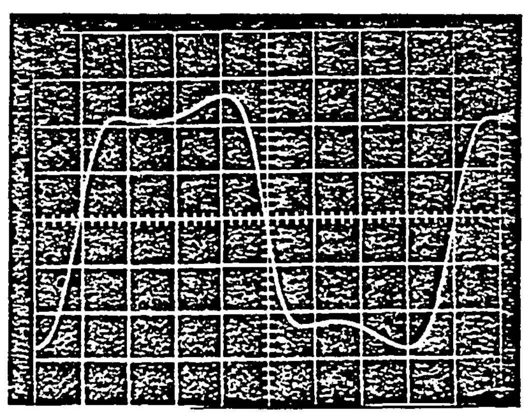


Figure 50. Voltage across transformer (V_T) (variac output 135 VRMS, C = 15 μF, scale 50 V/Division)

Experiment No. 4:

R = 9.02 Ω
L = 316 mH
C_X= 20.04 μF

Variac Output Voltage (Volts-RMS)	Voltage Across Voltage (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	21.5	13.5	0.09
20	35.2	24.9	0.16
25	44.5	34.5	0.23
30	54.8	41	0.292
35	65	48	0.34
40	73.7	52.8	0.377
45	80.7	54.8	0.391
50	85	55	0.392
55	88.8	54	0.383
60	90.5	52	0.369
65	94	50.5	0.353
70	95.8	48.5	0.338
75	97.6	46.2	0.32
80	99	45	0.309
85	101	43	0.292
90	102.7	42	0.28
95	104	41	0.27
100	105	40.9	0.262
105	106.4	41	0.258
110	107.4	41.2	0.256
115	108.7	42.1	0.26
120	110	43.5	0.265
125	111	45.8	0.275
130	111.9	48.2	0.29
135	112.8	51	0.301
138	113.3	52.6	0.313

No Ferroresonance

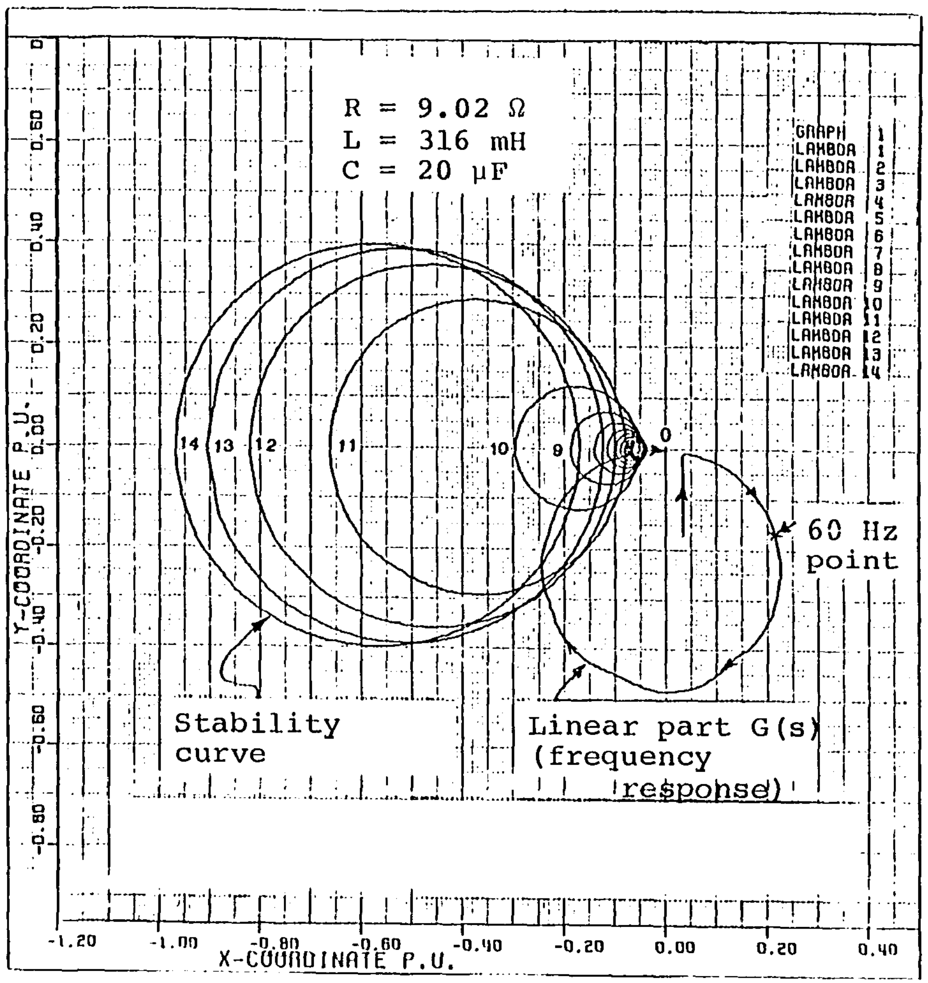


Figure 51. Stability and frequency response curves for pi circuit of Experiment No. 4

Experiment No. 4

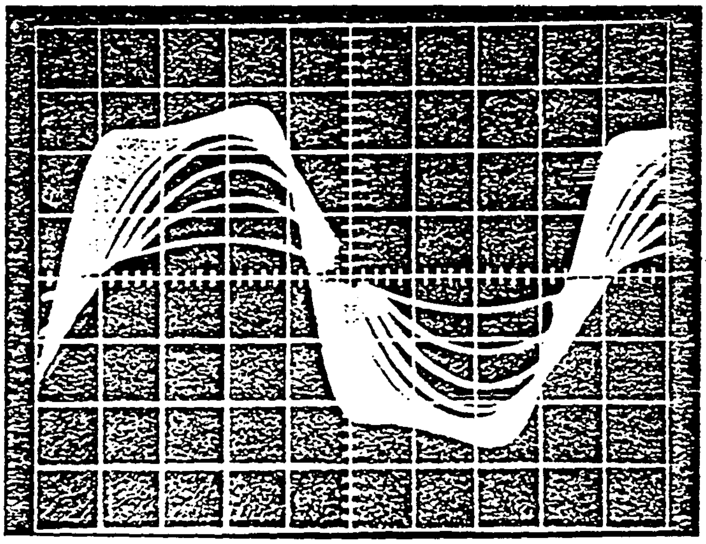


Figure 52. Voltage across transformer (V_T) (variac output between 15 V - 135 VRMS, voltage increment of 10 VRMS, C = 20 μF, scale 50 V/Division)

Experiment No. 4

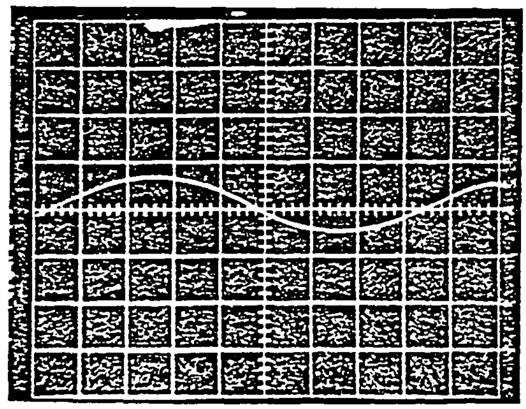


Figure 53. Voltage across transformer (V_T) (variac output 15 VRMS, C = 20 μF, scale 50 V/Division)

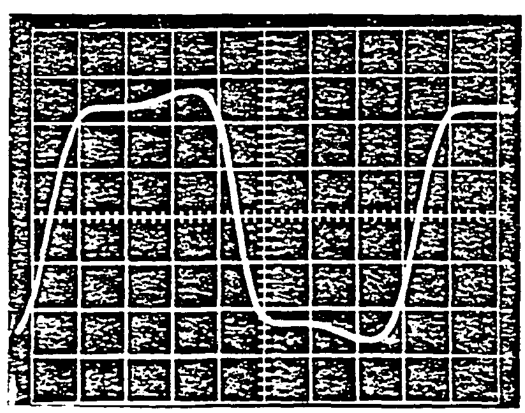


Figure 54. Voltage across transformer (V_T) (variac output 115 VRMS, C = 20 μF, scale 50 V/Division)

Experiment No. 5:

R = 9.02 Ω
L = 316 mH
C_X= 25.04 μF

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	29	27.5	0.19
20	43	45	0.3
25	56	55	0.39
30	69	67.2	0.478
35	80	75	0.53
40	87.8	76	0.537
45	92.2	74	0.52
50	95.2	70.8	0.5
55	98	67.4	0.475
60	100	64.2	0.452
65	102	61.2	0.429
70	103.9	58.2	0.408
75	105	55.4	0.387
80	106.4	53	0.363
85	108	50.8	0.341
90	109	49	0.327
95	110.2	47	0.31
100	111.1	45.6	0.3
105	112.1	45	0.29
110	113	44.8	0.283
115	114	44.6	0.28
120	114.6	44.2	0.28
125	115.5	46.2	0.282
130	116.2	48.3	0.291
135	117	50.2	0.302
138	117.5	52.1	0.312

No Ferroresonance

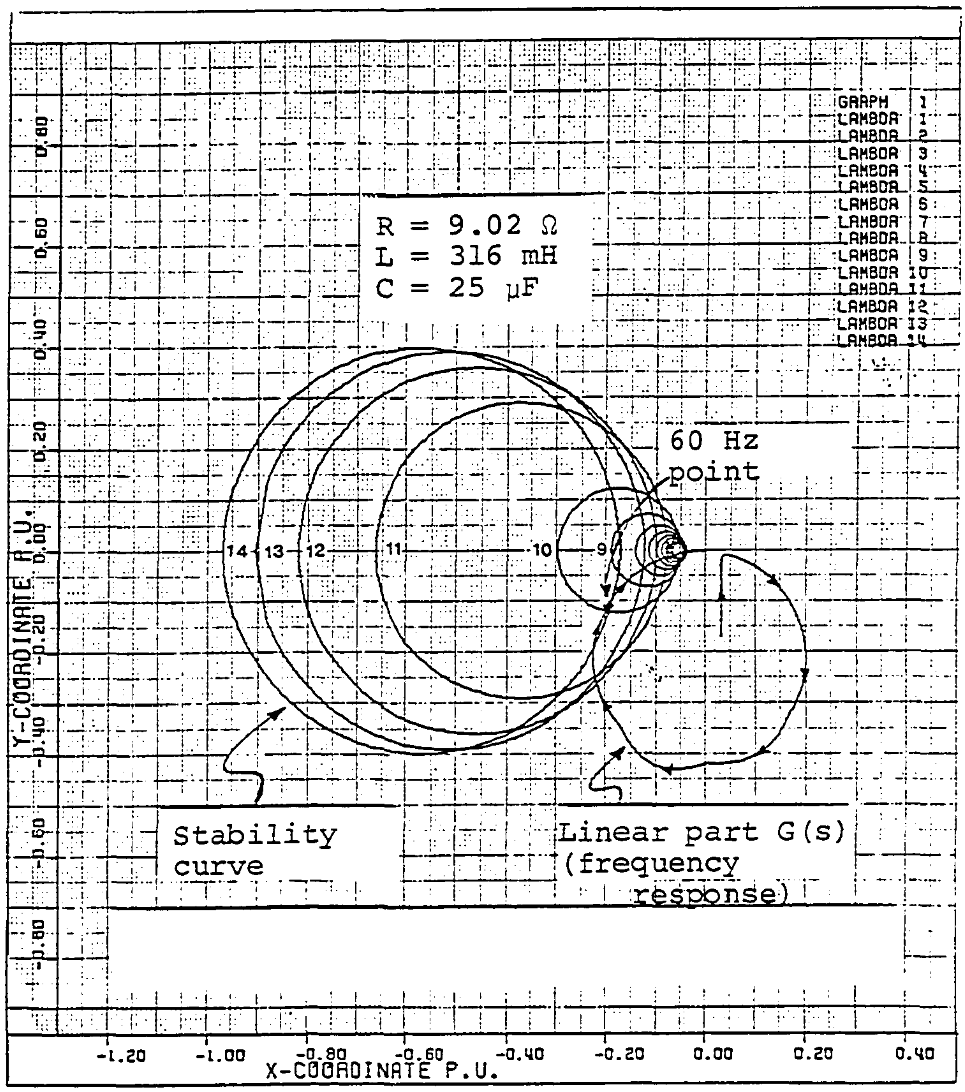


Figure 55. Stability and frequency response curves for pi circuit of Experiment No. 5

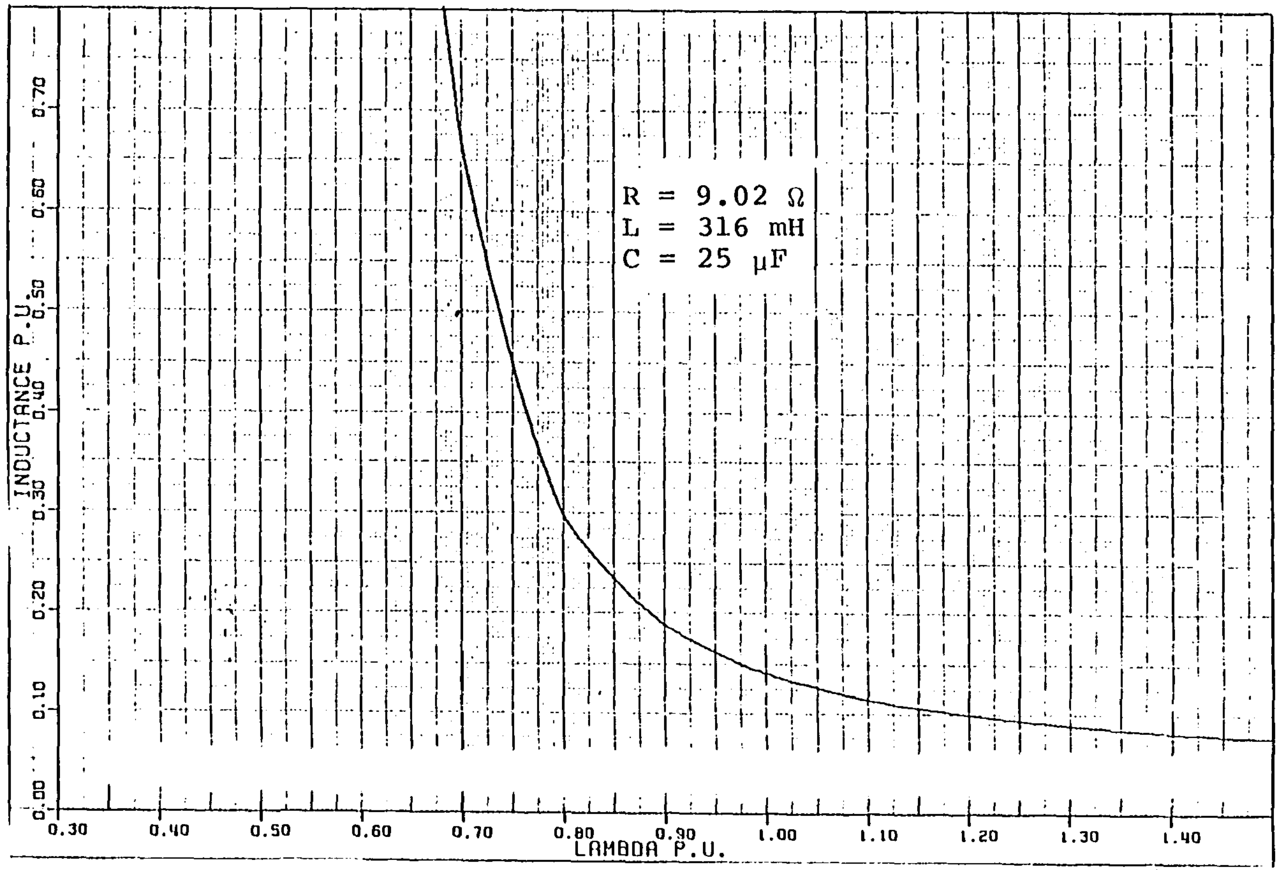


Figure 56. Inductance vs. lambda for pi circuit of Experiment No. 5

Experiment No. 5

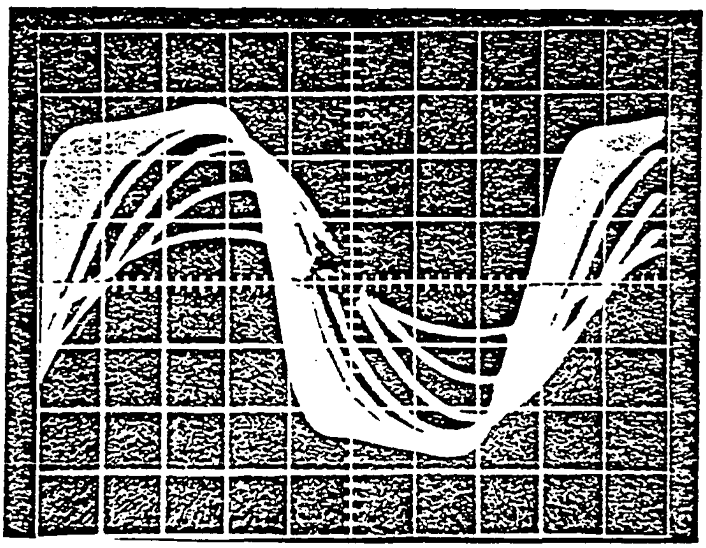


Figure 57. Voltage across transformer (V_T) (variac output between 15 V - 135 VRMS, voltage increments of 10 VRMS, C = 25 μF, scale 50 V/Division)

Experiment No. 5

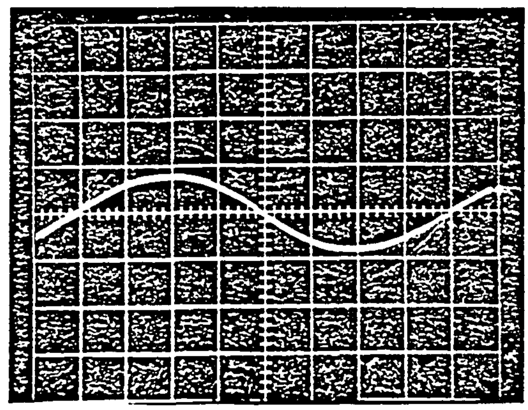


Figure 58. Voltage across transformer (V_T) (variac output 15 VRMS, C = 25 μF, scale 50 V/Division)

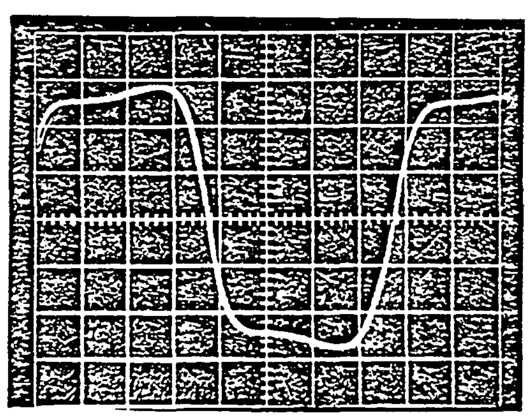


Figure 59. Voltage across transformer (V_T) (variac output 135 VRMS, C = 25 μF, scale 50 V/Division)

Experiment No. 6:

$R = 9.02\ \Omega$
$L = 316$ mH
$C_X = 29.69\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)	
15	33	37	0.257	
20	42	49.5	0.355	
25	51.5	61.7	0.45	
30	63.2	79	0.55	Increase
35	78.5	94	0.658	
40	96.4	92	0.651	
45	100.7	87	0.612	
50	103.5	82	0.574	
55	105.5	78.8	0.543	
60	107	73.7	0.516	
65	108.5	70	0.489	
70	110	66.8	0.463	
75	110.3	63.3	0.437	Decrease
80	112.3	60.5	0.413	
85	113.3	57.2	0.39	
90	114.2	55	0.369	
95	115.0	52.3	0.35	
100	116.0	51	0.331	
105	116.7	49.2	0.317	
110	117.4	48.2	0.307	
115	118.2	48	0.301	
120	118.8	47.8	0.299	
125	119.5	48	0.3	
130	120.1	48.4	0.301	
135	120.7	50	0.307	
138	121.2	52	0.317	

There is a slight jump at 35V, but it cannot be seen on the oscilloscope CRT.

Experiment No. 7:

R = 9.02 Ω
L = 316 mH
C_X = 35.02 μF

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)	
15	25.6	34.3	0.24	
20	34	47	0.332	
25	40.3	57	0.41	
30	46	68.5	0.489	
35	53.7	80	0.568	Increase
40	62.4	93	0.656	
45	70.8	104.8	0.737	
46	72.8	107.7	0.757	
47	75	110.5	0.777	
48	77.7	113.5	0.798	
49 F.R. occurs	109	95.5	0.668	
50	109.3	94.3	0.66	
55	111.4	88	0.579	
60	113.1	82.9	0.579	
65	114.1	78.4	0.544	
70	115	74.2	0.514	
75	116	70.3	0.488	
80	117	67	0.46	
85	117.8	63.2	0.433	
90	118.5	60.8	0.408	
95	119.2	57.8	0.385	Decrease
100	120	55	0.363	
105	120.6	53	0.345	
110	121.2	52	0.332	
115	121.7	51	0.321	
120	122.2	50.5	0.315	
125	122.8	50.1	0.312	
130	123.2	50.2	0.312	
135	123.8	51	0.317	
137	124.2	52	0.32	

Experiment No. 7

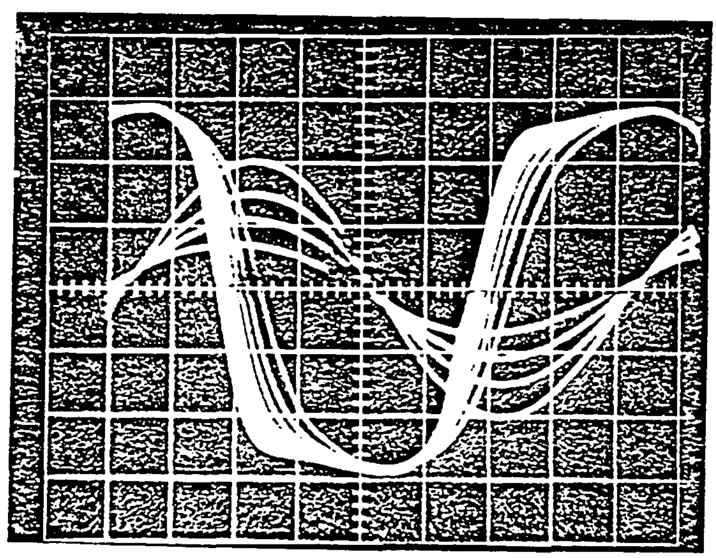


Figure 60. Voltage across transformer (V_T) linear and nonlinear (variac output between 15 V - 135 VRMS, voltage increment of 15 VRMS, C = 35 μF, scale V/Division)

Experiment No. 7

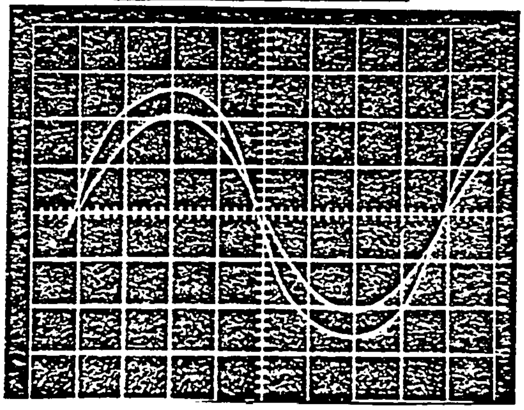


Figure 61. Voltage across transformer linear mode (80 VRMS) (variac output voltage 48 VRMS, C = 35 μF). Voltage across transformer nonlinear mode (108.7 VRMS, variac output voltage 48.5 VRMS, C = 35 μF, scale 50 V/Division)

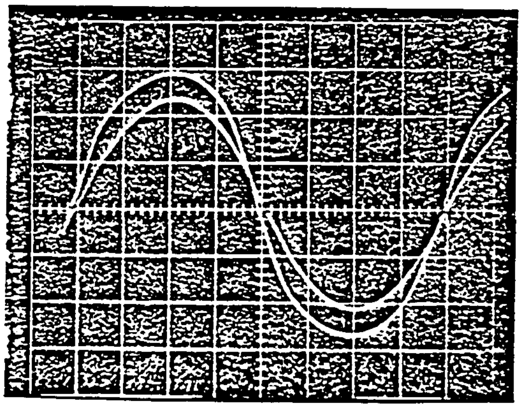


Figure 62. Voltage across transformer linear mode (80 VRMS) (variac output voltage 48 VRMS, C = 35 μF). Voltage across transformer second mode (108.7 VRMS, variac output voltage 48.5 VRMS, C = 35 μF, scale 50 V/Division)

Experiment No. 7

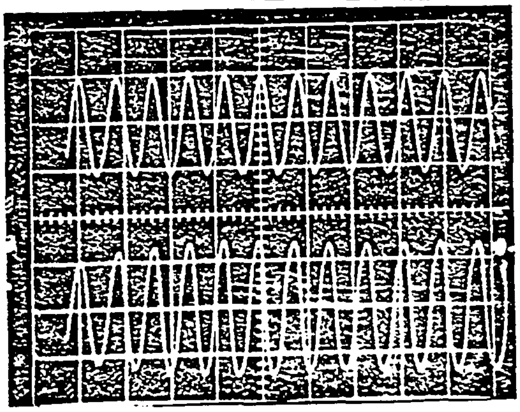


Figure 63. Voltage across transformer linear mode (upper), voltage across transformer nonlinear mode (lower) (voltage period 20 ms/division, C = 35 μF, scale 50 V/Division)

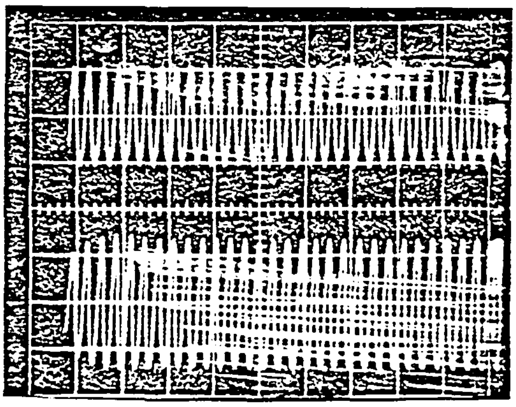


Figure 64. Voltage across transformer linear mode (upper), voltage across transformer nonlinear mode (lower) (voltage period 50 ms/division, C = 35 μF, scale 50 V/Division)

Experiment No. 7

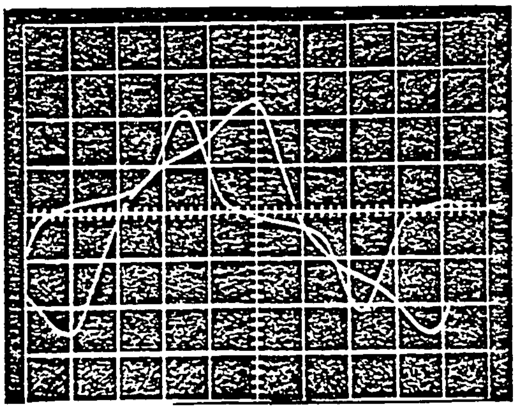


Figure 65. Transformer current (I_T) just before and after ferroresonance (variac output 48 VRMS, C = 35 μF, scale before F.R. 0.1 V/Division, scale after F.R. 0.5 V/Division, shunt 50 MV/A)

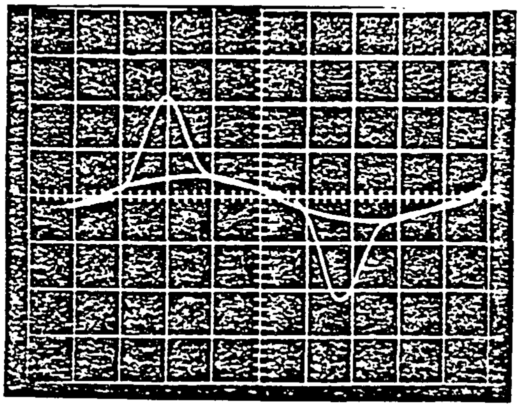


Figure 66. Transformer current (I_T) just before and after loading transformer with R = 150 Ω, scale 0.5 V/Division, C = 35 μF, shunt 50 MV/A)

Experiment No. 8:

$R = 9.02\ \Omega$
$L = 316$ mH
$C_X = 40.05\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Trans. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)	
15	19.8	31	0.215	Increase
20	26	42	0.295	
25	31.7	52	0.37	
30	36.5	61.6	0.435	
35	41.5	70.8	0.5	
40	46.5	80	0.572	
45	52	90	0.64	
50	57.5	100	0.71	
55	63.2	112	0.779	
60	70	123	0.855	
65	78	134	0.938	
66 F.R. occurs	80.5	138	0.96	
67	120	83	0.575	
70	120	80	0.556	
75	121	77	0.524	
80	121.3	73.5	0.502	
85	122	70	0.473	Decrease
90	122.5	66	0.449	
95	123.2	63	0.42	
100	124.0	60	0.398	
105	124.3	58	0.377	
110	124.8	56	0.36	
115	125.2	54	0.348	
120	126.0	53.8	0.34	
125	126.2	53.2	0.33	
130	126.8	53	0.33	
135	127.2	53.3	0.33	
138	127.5	54	0.331	

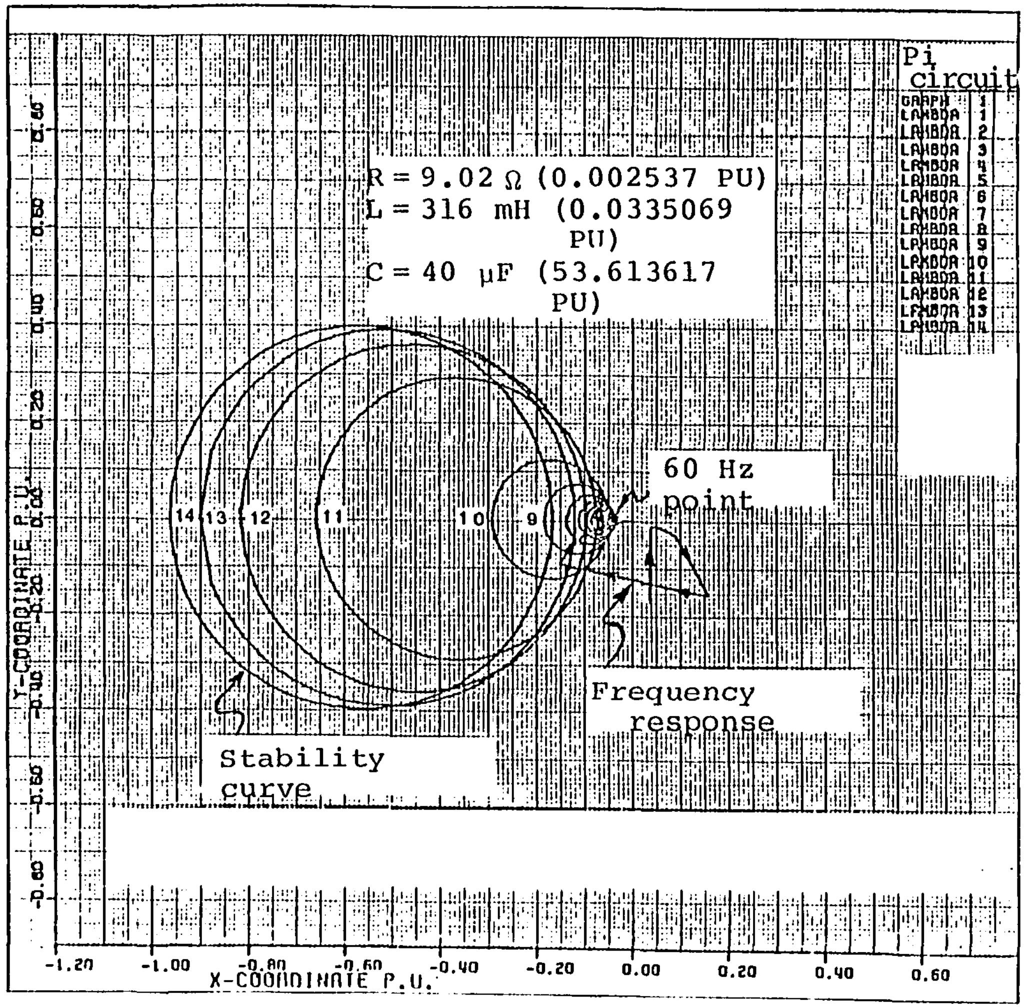


Figure 67. Stability and frequency response curves for pi circuit of Experiment No. 8

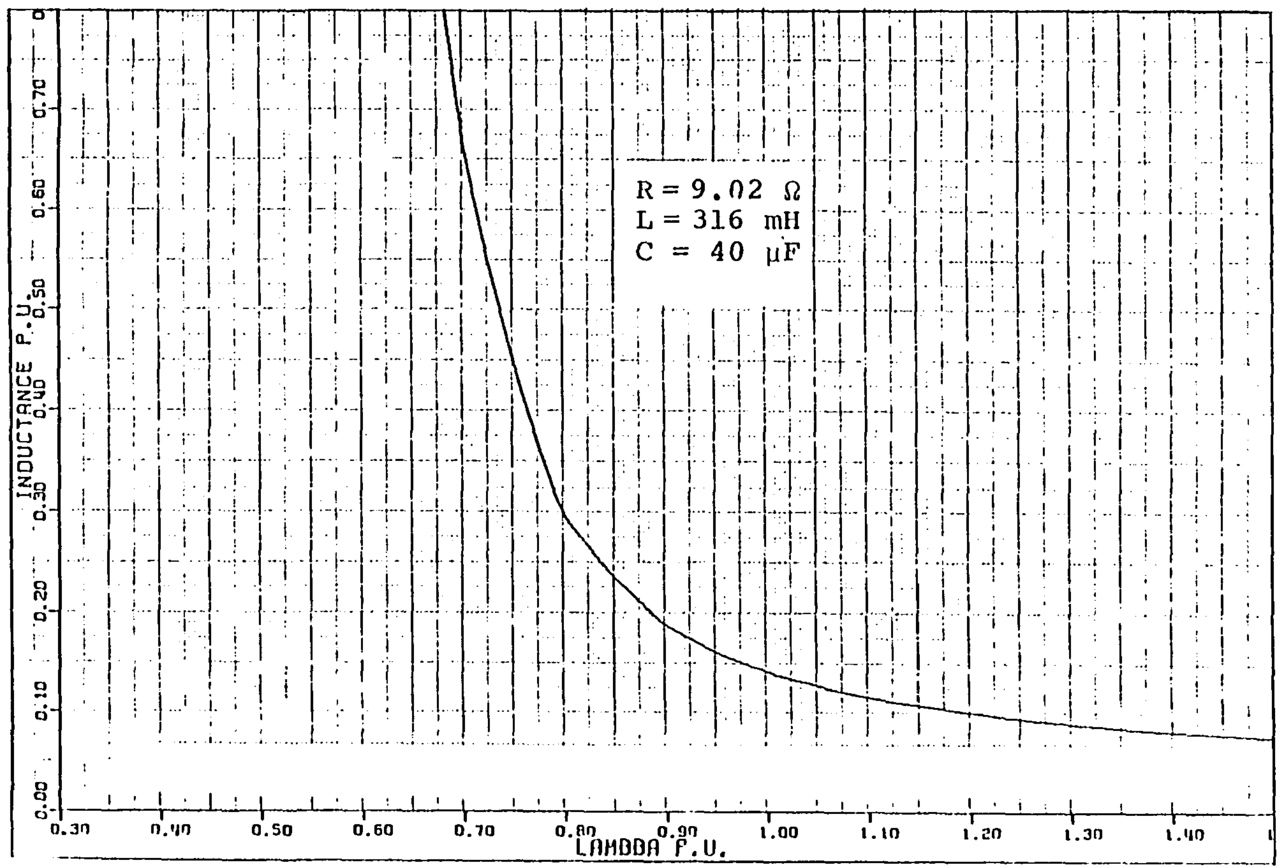


Figure 68. Inductance vs. lambda for pi circuit of Experiment No. 8

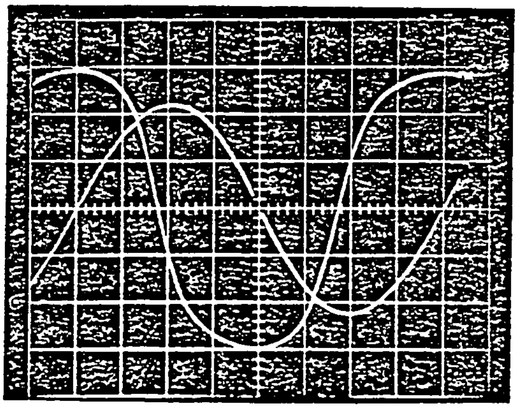


Figure 69. Voltage across transformer (V_T) just before and after ferroresonance (variac output 66 VRMS, C = 40 μF, scale 50 V/Division)

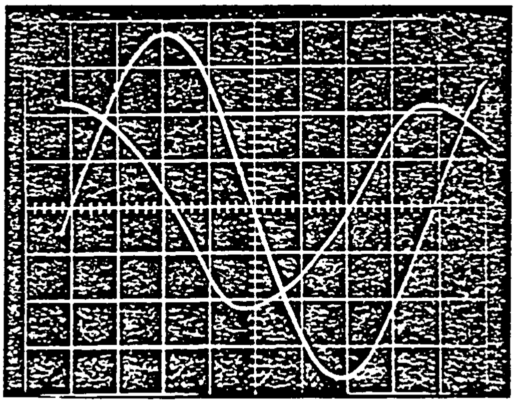


Figure 70. Transformer current (I_T) just before and after ferroresonance (variac output 66 VRMS, C = 40 μF, scale 0.02 V/Division, shunt 50 MV/A)

Experiment No. 8

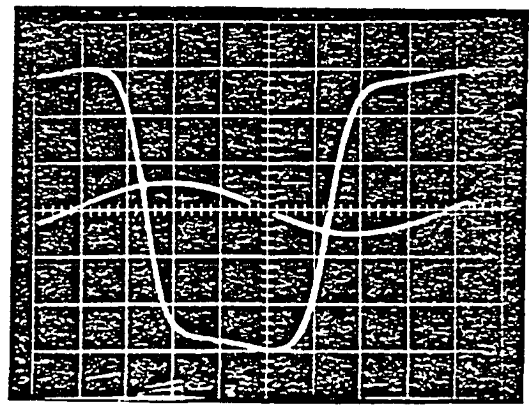


Figure 71. Voltage across transformer linear and nonlinear modes (variac output 15 V and 135 VRMS, C = 40 μF, scale 50 V/Division)

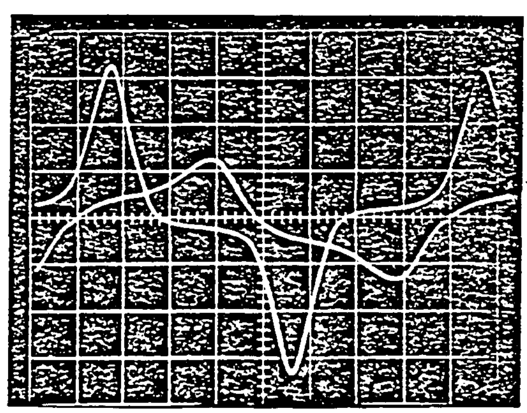


Figure 72. Transformer current linear mode scale 0.02 V/ Division, transformer current nonlinear mode (scale 0.05 V/Division, shunt 50 MV/A)

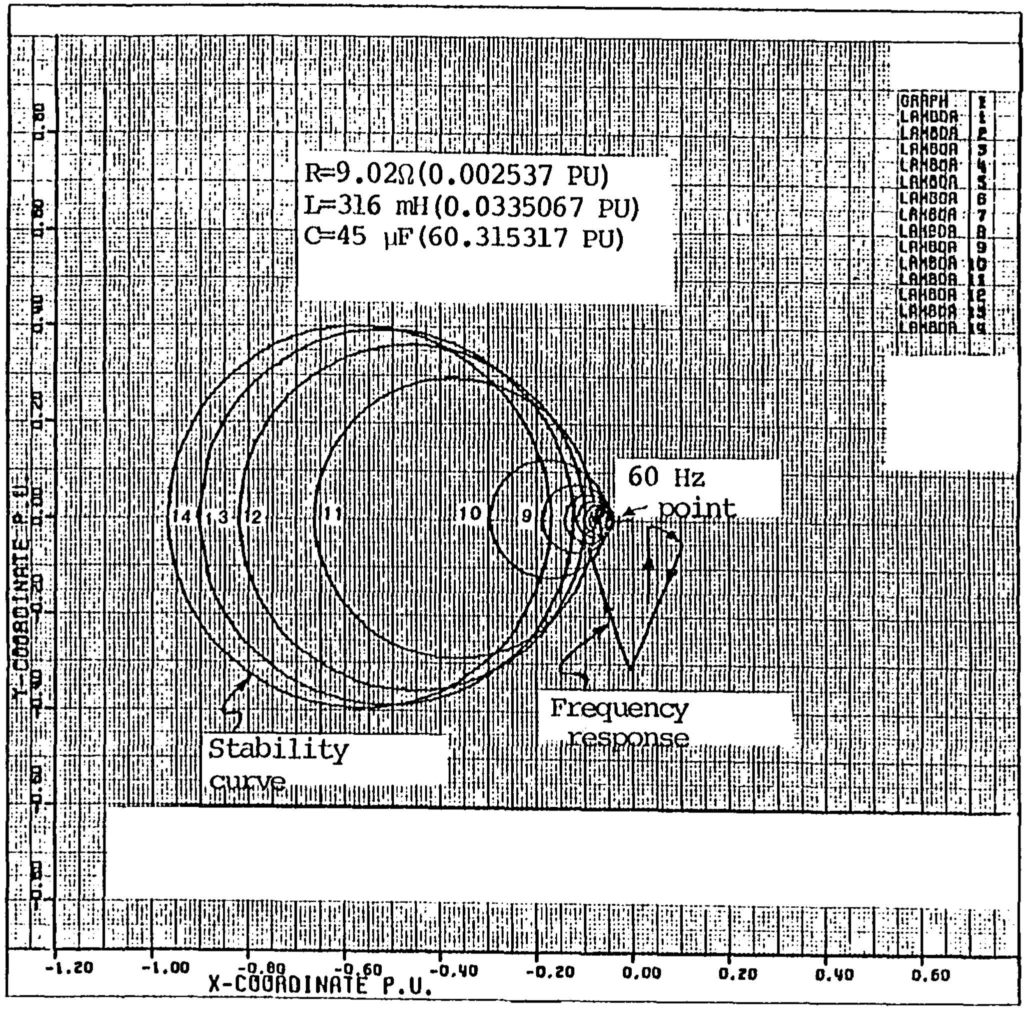


Figure / :. Stability and frequency response curves for the values shown

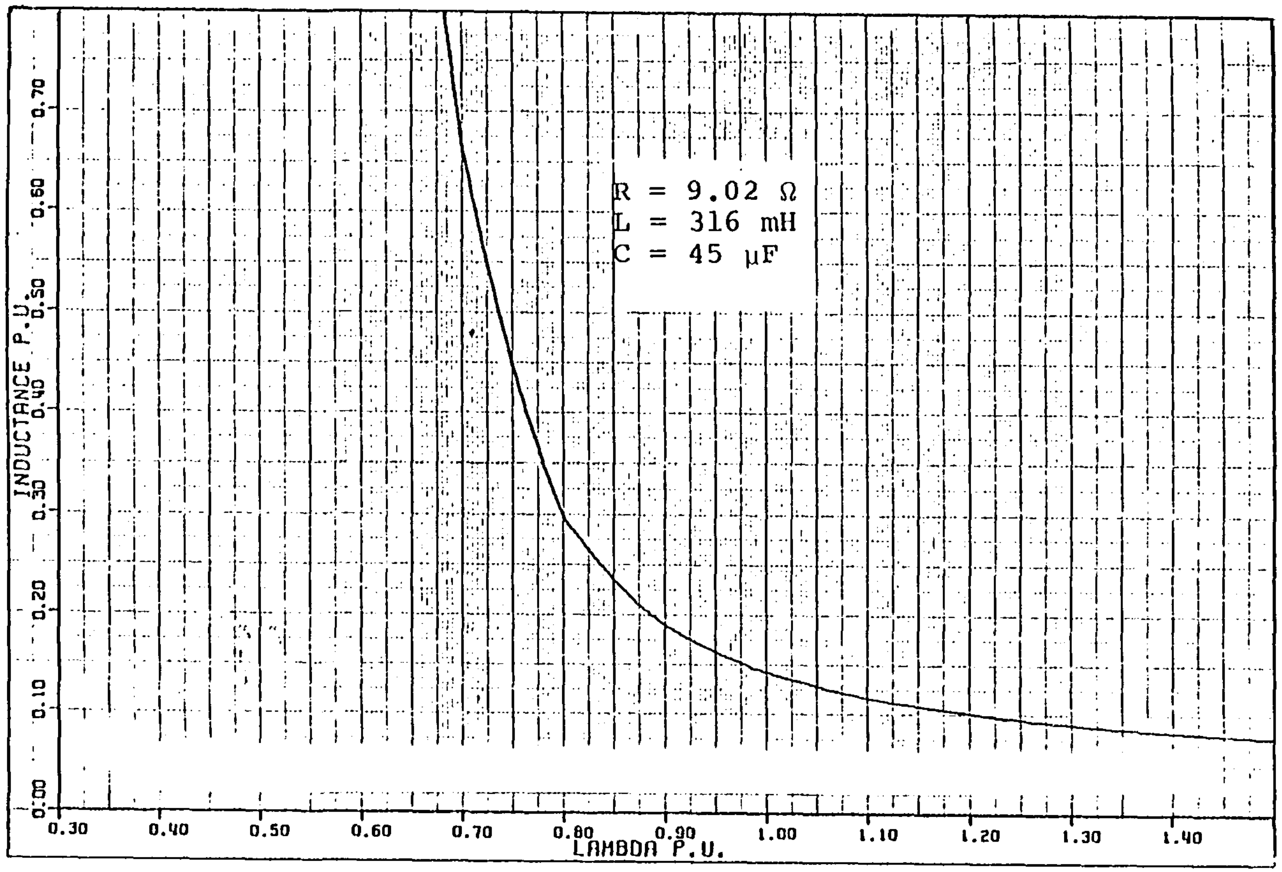


Figure 74. Inductance vs. lambda for the values shown

Experiment No. 9:

$R = 9.02\ \Omega$
$L = 316$ mH
$C_X = 54.9\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)	
15	13	26.5	0.19	
20	16.3	35	0.245	
25	19.3	42.7	0.32	
30	22	50	0.355	
35	25.3	57.4	0.41	
40	28.1	65.5	0.468	
45	30.8	73	0.517	
50	33.2	81.2	0.57	
55	36.2	89.2	0.628	
60	39	91.2	0.677	
65	42	103.8	0.729	
70	44.5	117.7	0.783	
75	47	118.2	0.829	Increase
80	50	126.0	0.883	
85	53.4	135.3	0.95	
90	56	142.5	0.99	
95	59.8	150	1.042	
100	62.7	159.4	1.102	
105	65.7	163.2	1.158	
110	69	175	1.212	
115	72.2	183.7	1.262	
120	76	197	1.325	
125	80.3	201	1.384	
130	86.2	210.3	1.455	F.R. occurs
132	134.5	60.7	0.39	Decrease
135	135	60	0.39	
138	135	60	0.39	

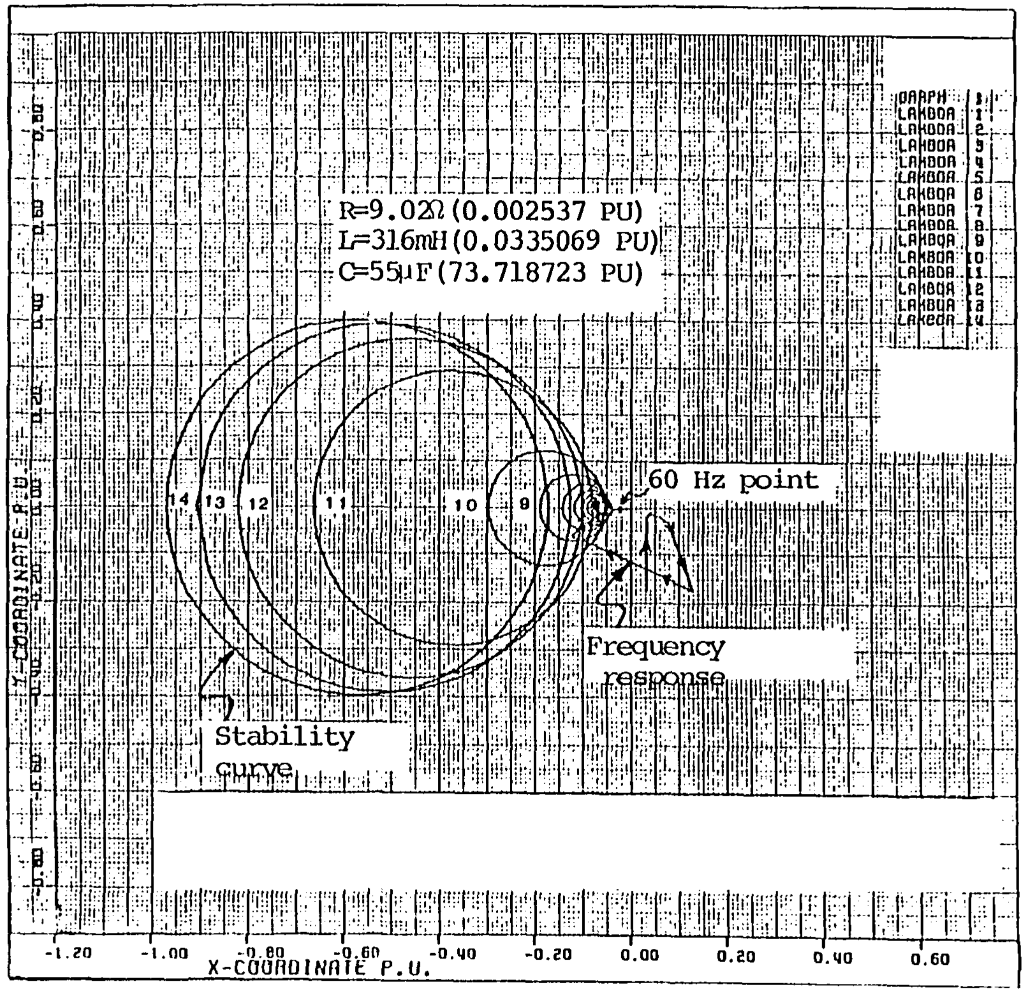


Figure 75. Stability and frequency response curves for pi circuit of Experiment No. 9

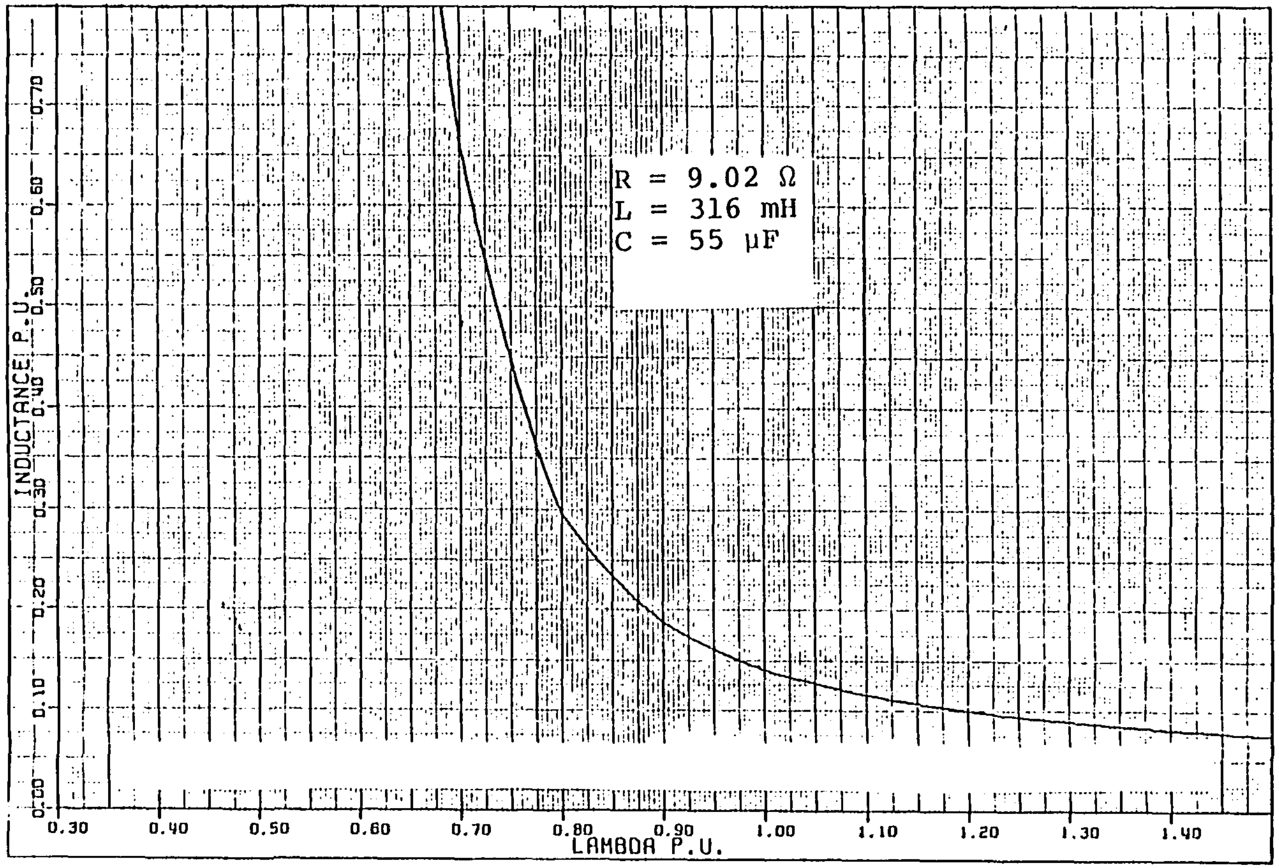


Figure 76. Inductance vs. lambda for pi circuit of Experiment No. 9

Experiment No. 9

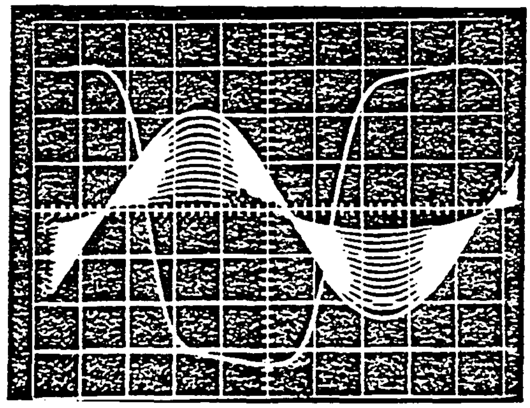


Figure 77. Voltage across transformer linear and nonlinear modes (variac output between 15 V - 135 VRMS, voltage increments of 10 VRMS, C = 35 μF, scale 50 V/Division)

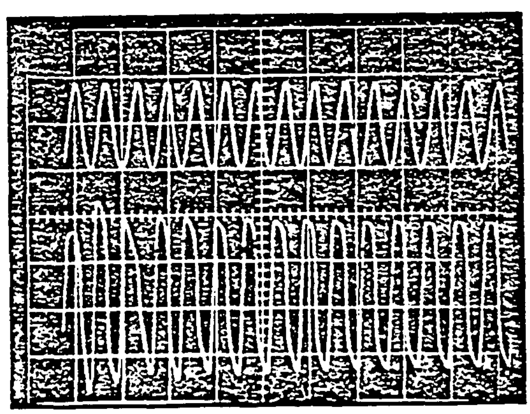


Figure 78. Voltage across transformer linear mode (upper), voltage across transformer nonlinear mode (lower) (C = 55 μF, scale 50 V/Division)

Experiment No. 9

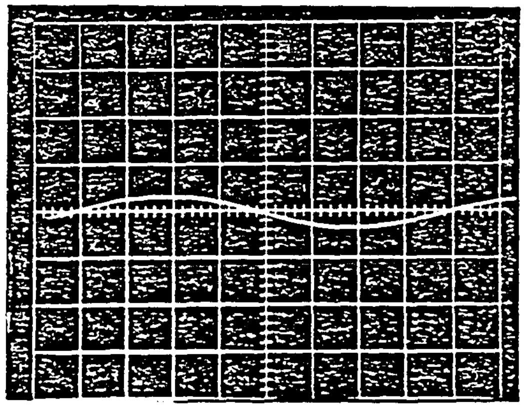


Figure 79. Voltage across transformer (V_T), variac output 15 VRMS, C = 55 μF, scale 50 V/Division)

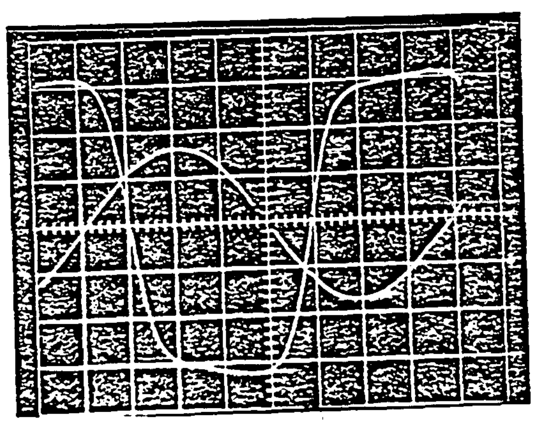


Figure 80. Voltage across transformer linear mode (Sinusoidal), voltage across transformer nonlinear mode (distorted) (C = 55 μF, scale 50 V/Division)

Experiment No.

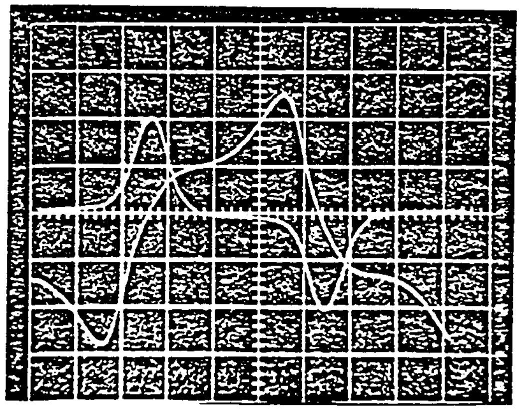


Figure 81. Transformer current (I_T) just before and after ferroresonance, variac output 130 VRMS, C = 55 μF, scale before F.R. 0.01 V/Division, scale after F.R. 0.2 V/Division, shunt 50 MV/A)

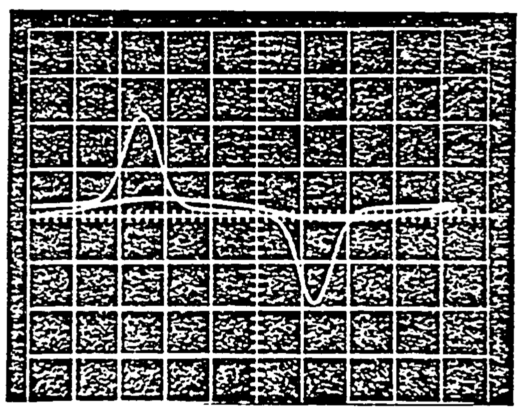


Figure 82. Transformer current (I_T) just before and after loading transformer with R = 150 Ω, scale 0.2/ division, C = 55 μF, shunt 50 MV/A

Experiment No. 10:

R = 9.02 Ω
L = 316 mH
C_X= 64.7 μF

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	9	22.2	0.15
20	11.8	30.7	0.215
25	14.1	38	0.267
30	16.5	45	0.315
35	19	51.8	0.369
40	21.3	59.5	0.42
45	23.3	66.5	0.467
50	25.7	73.2	0.515
55	28	81	0.569
60	30	87.7	0.617
65	32.3	94.4	0.664
70	34.4	101.3	0.712
75	36.5	108.2	0.758
80	38.4	114.5	0.805
85	40.2	121.5	0.852
90	42.2	128.5	0.9
95	44.2	135.8	0.948
100	47	145.5	1.01
105	49.5	152	1.05
110	51.7	159.8	1.1
115	53.7	166	1.15
120	56	173.5	1.2
125	58.4	187	1.225
130	60.8	187.5	1.295
135	63	195	1.34
138	65	201	1.382

No Ferroresonance

Experiment No. 10

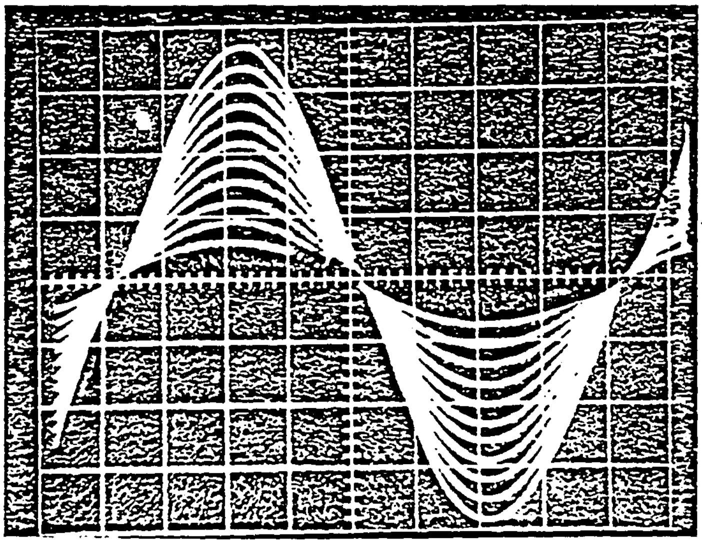


Figure 83. Voltage across transformer (V_T) (variac output between 15 V - 135 VRMS, voltage increment of 10 VRMS, C = 65 μF, scale 100 V/Division)

B. Effect of Pi-Circuit Input Capacitance on Ferroresonance

The following four experiments showed that the input capacitance of the transmission line has no effect on the magnitudes of critical voltages.

R = 9.02
L = 316 mH
C_1 = 35 mH
C_2 = 10 μF

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	25	3	0.21
20	33.5	46	0.27
25	40	56.3	0.339
30	45.8	67.8	0.402
35	53	79.7	0.466
40	61.7	92.6	0.538
45	70	104	0.61
46	72	107	0.626
47	74.3	110	0.644
48	77.3	113	0.664
49	80.8	116.5	0.688
49.5 F.R. occurs	109	96.3	0.723
50	109.2	95.8	0.7215
55	111.2	89.8	0.708
60	113	84.2	0.695
65	114.1	79.5	0.681
70	115.2	75	0.669
75	116.4	70.5	0.655
80	117	67.2	0.643
85	118	63.5	0.629
90	118.8	60.5	0.616
95	119.2	57.4	0.602
100	120	55	0.589
105	120.8	53.1	0.573
110	121.2	52	0.561
115	121.8	51.2	0.547
120	122.5	51	0.535
125	123	51	0.522
130	123.7	51.5	0.51
135	124	52.6	0.498
138	124.5	53.7	0.488

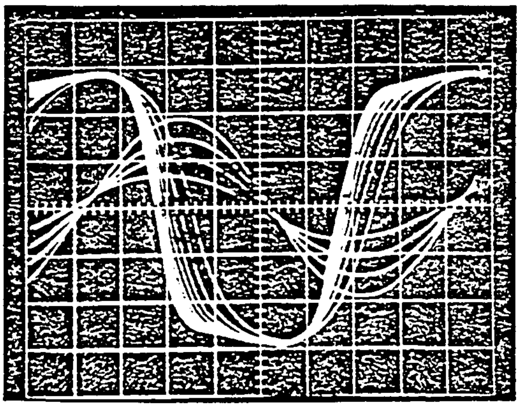


Figure 84. Voltage across transformer (V_T), linear and nonlinear modes (variac output between 15 V - 135 VRMS, C_1 = 35 μF, C_2 = 10 μF, scale 50 V/Division)

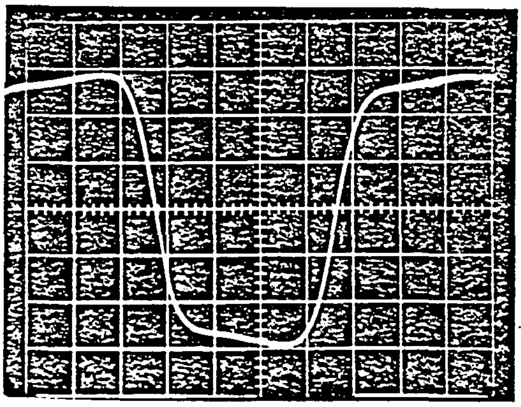


Figure 85. Voltage across transformer (V_T), nonlinear mode (variac output 135 VRMS, C_1 = 35 μF, C_2 = 10 μF, scale 50 V/Division)

R = 9.02 Ω
L = 316 mH
C_1 = 35 μF
C_2 = 30 μF

Variac Output Voltage (Volts-RMS)		Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15		25	34.5	0.165
20		33.5	46	0.215
25		40	56.8	0.258
30		46.5	68.7	0.3
35		53.8	80.7	0.339
40		62	93	0.388
45		70.8	105.2	0.443
46		73	108.2	0.458
47		75.5	111.2	0.476
48	F.R. occurs	78.8	114.5	0.497
49		109	96.5	0.92
50		109.2	95.2	0.93
55		111.5	89.0	0.985
60		113	83.8	1.02
65		114.2	79	1.052
70		115.2	75	1.088
75		116.5	70.5	1.12
80		117.4	66.8	1.15
85		118	63.1	1.18
90		118.8	60.2	1.208
95		119.5	57.3	1.233
100		120	55	1.26
105		120.7	53.1	1.285
110		121.2	52	1.31
115		122	51.4	1.335
120		122.5	51	1.36
125		123	51.1	1.382
130		123.7	52	1.408
135		124.2	53	1.43
138		124.5	54	1.43

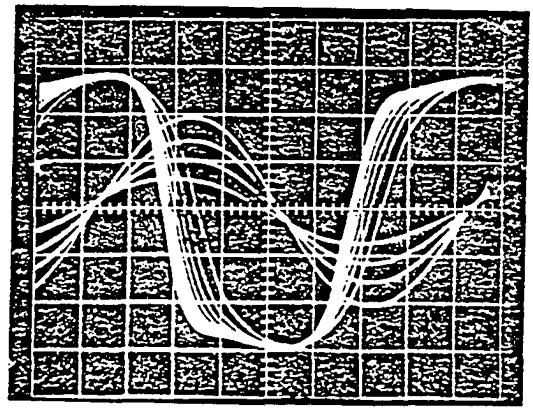


Figure 86. Voltage across transformer (V_T), linear and nonlinear modes (variac output between 15 V - 135 VRMS, C_1 = 35 μF, C_2 = 30 μF, scale 50 V/division)

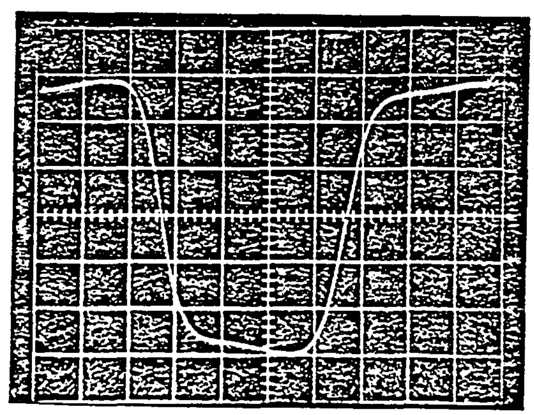


Figure 87. Voltage across transformer (V_T), nonlinear mode (variac output 135 VRMS, C_1 = 35 μF, C_2 = 30 μF, scale 50 V/division)

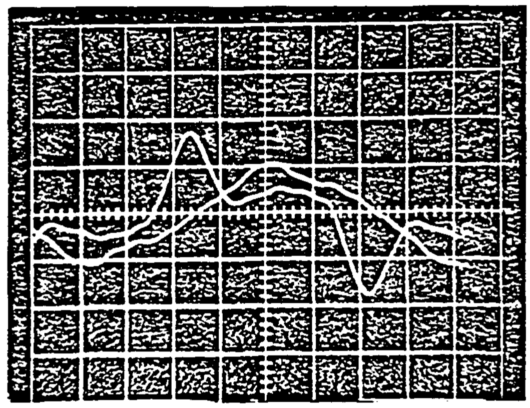


Figure 88. Transformer current (I_T) just before and after F.R. (variac output 48 VRMS, C_1 = 35 μF, C_2 = 30 μF, scale 0.5 V/division, shunt 50 MV/A)

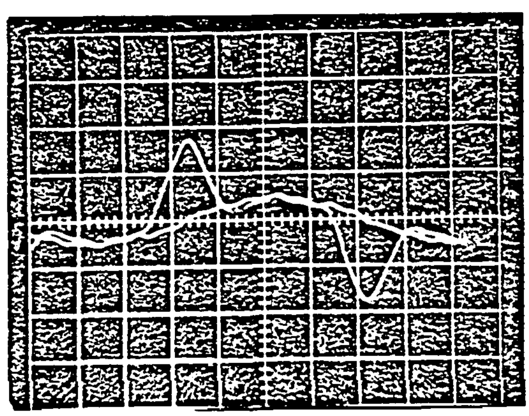


Figure 89. Transformer current (I_T) just before and after loading with 150 at F.R. (variac output 48 VRMS, C_1 = 35 μF, C_2 = 30 μF, scale 0.5 V/division, shunt 50 MV/A)

$R = 9.02\ \Omega$
$L = 316\ mH$
$C_1 = 55\ \mu F$
$C_2 = 10\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	12	25.7	0.12
20	15.5	34.5	0.17
25	18.5	42.3	0.21
30	21.5	49.9	0.25
35	24.5	57	0.285
40	27.8	65.3	0.322
45	30.3	72.8	0.359
50	33.2	81	0.394
55	36.2	89	0.432
60	39	96.6	0.468
65	42	104	0.502
70	44.5	111.1	0.533
75	47.5	119.2	0.57
80	50.1	126.4	0.604
85	53.2	135	0.642
90	56.2	142.3	0.675
95	59.5	149	0.705
100	62.1	158	0.74
105	65.1	166	0.778
110	68.4	174	0.814
115	71.7	182	0.85
120	75	190	0.889
125	79.3	199	0.932
130 F.R. occurs	84	208	0.978
133	135	61.5	0.6
135	135	61	0.594
138	135.2	60.8	0.582

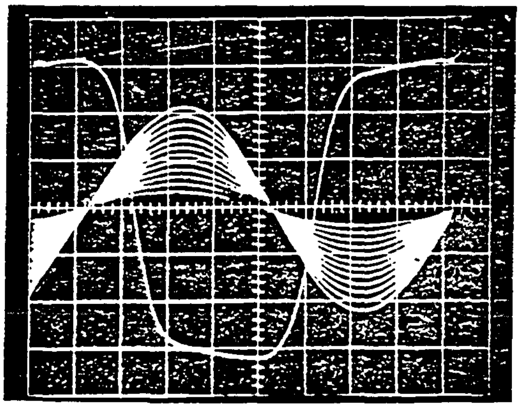


Figure 90. Voltage across transformer (V_T), linear and nonlinear modes (variac output between 15 V - 135 VRMS, voltage increment of 10 VRMS, C_1 = 55 μF, C_2 = 10 μF, scale 50 V/division)

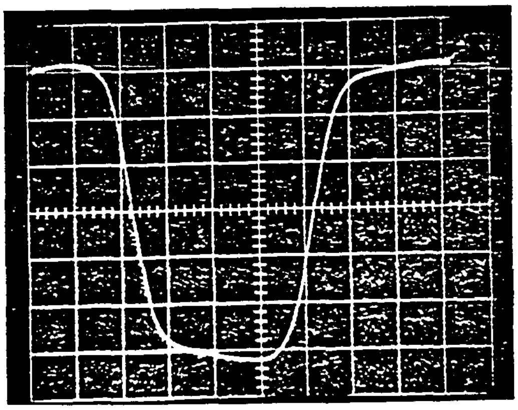


Figure 91. Voltage across transformer (V_T), second mode (variac output 135 VRMS, C_1 = 55 μF, C_2 = 10 μF, scale 50 V/division)

$R = 9.02\ \Omega$
$L = 316$ mH
$C_1 = 55\ \mu F$
$C_2 = 50\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Total Current (A-RMS)
15	12	25.7	0.10
20	15.8	35.2	0.14
25	18.8	42.1	0.20
30	22	50.8	0.235
35	24.9	58	0.28
40	28	66.6	0.321
45	31	74	0.363
50	33.8	82.5	0.41
55	37	90.8	0.455
60	39.8	98.5	0.5
65	43	105.7	0.545
70	45	113.2	0.589
75	48.4	121.3	0.633
80	51.2	129.3	0.678
85	54.1	137.2	0.721
90	57.2	145	0.765
95	60.2	153	0.79
100	63.2	161.5	0.834
105	66.6	170	0.878
110	70	178	0.917
115	73.2	186	0.96
120	77.2	194.5	1.0
125	81.3	203	1.02
130	88	213	1.05
132 F.R. occurs	134.5	61	2.0

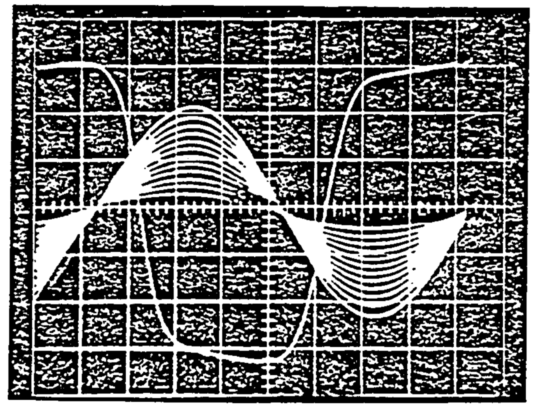


Figure 92. Voltage across transformer (V_T), linear and nonlinear modes (variac output between 75 V - 135 VRMS, C_1 = 55 μF, C_2 = 50 F, scale 50 V/division)

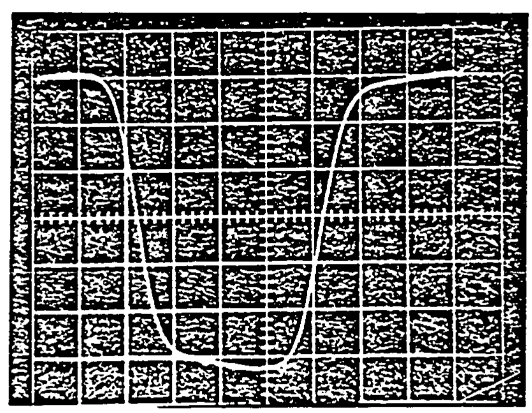


Figure 93. Voltage across transformer (V_T), nonlinear mode (variac output 135 VRMS, C_1 = 55 μF, C_2 = 50 μF, scale 50 V/division)

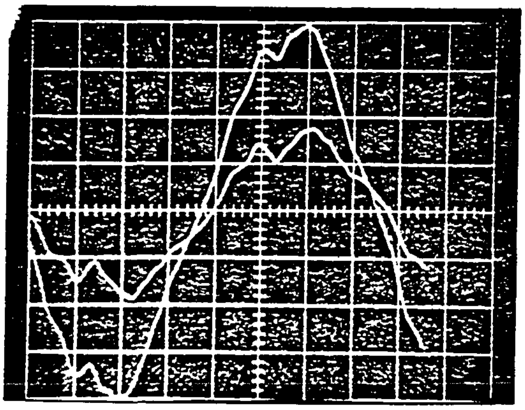


Figure 94. Transformer current (I_T) just before and after F.R. (variac output 130 VRMS, C_1 = 55 μF, C_2 = 50 μF, scale 0.5 V/division, shunt 50 MV/A)

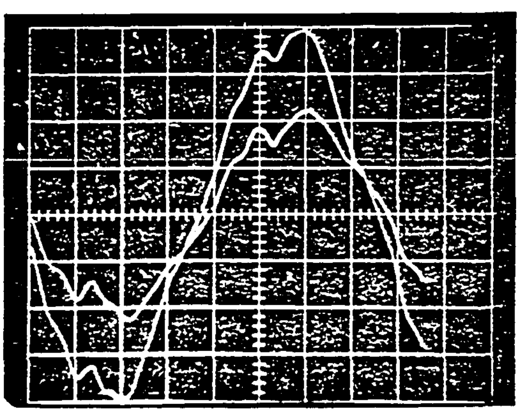


Figure 95. Transformer current (I_T) just before and after loading with 150 Ω at F.R. (variac output 130 VRMS, C_1 = 55 μF, C_2 = 50 μF, scale 0.5 V/division, shunt 50 MV/A)

C. Switching Modes by a Circuit Breaker

1. Pi circuit

LM = Linear Mode R = 9.02 Ω
NLM = Nonlinear Mode L = 316 mH
C = 35 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	40	62.5		
LM	40	62.5	>100	>100
LM	42	67.3		
NLM	42	103	1	1
LM	42	67	1	1
NLM	42	103	2	2
LM	42	66.8	1	1
NLM	42.2	102.8	3	3
LM	42	66.5	1	1
NLM	42	102.8	51	51
LM	42	67.5	1	1
NLM	42	103.2	5	5
LM	42	66.7	1	1
NLM	42	102.7	42	42
LM	42	66.5	3	3
NLM	42	102.8	11	11
LM	42	66.5	1	1
LM	42	66.5	600	600
LM	44	70		
NLM	44	106	1	1
LM	44	70	2	2
NLM	44	106	2	2
LM	44	70	1	1
NLM	44	106	1	1
LM	44	71	1	1
NLM	44	106.2	1	1
LM	44	71	4	4
NLM	44	106.1	1	1

LM = Linear Mode
NLM = Nonlinear Mode

R = 9.02 Ω
L = 316 mH
C = 35 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	44	71	1	1
NLM	44	106.3	1	1
LM	44	71	3	3
NLM	44	106.2	3	3
LM	44	71	1	1
NLM	44	106.2	1	1
LM	44	70	1	1
NLM	44	106	2	2
LM	44	70	1	1
NLM	44	106.2	1	1
LM	44	70	3	3
LM	44	70		
NLM	44	106.2	2	2
LM	44	70	2	2
NLM	44	106.1	5	5
LM	44	71	1	1
NLM	44	106.2	2	2
LM	44	71	1	1
NLM	44	106.3	1	1
LM	44	71	3	3
NLM	44	106.2	3	3
LM	44	71	2	2
NLM	44	106.2	3	3
LM	44	71	2	2
NLM	44	106.3	1	1
LM	44	71	1	1
NLM	44	106.3	1	1
LM	44	71	1	1
NLM	44	106.2	1	1
LM	44	71	4	4
NLM	44	106.2	2	2
LM	44	71	1	1

LM = Linear Mode
NLM = Nonlinear Mode

R = 9.02 Ω
L = 316 mH
C = 35 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	44	71		
NLM	44	106.2	1	1
LM	44	71	1	1
NLM	44	106.3	1	1
LM	44	71	3	3
NLM	44	106.2	1	1
LM	44	71.1	1	1
NLM	44	106.2	3	3
LM	44	71	1	1
NLM	44	106.2	2	2
LM	46	76.2		
NLM	46	107.8	1	1
LM	46	76	6	6
NLM	46	107.5	1	1
LM	46	76	1	1
NLM	46	107.8	1	1
LM	46	75.8	13	13
NLM	46	107.7	1	1
LM	46	75.7	8	8
NLM	46	107.6	2	2
NLM	46	107.6		
LM	46	76	12	12
NLM	46	107.8	1	1
LM	46	76	1	1
NLM	46	107.8	1	1
LM	46	76	3	3
NLM	46	107.8	1	1
LM	46	76	3	3
NLM	46	107.6	1	1
LM	46	75.7	4	4
NLM	46	107.8	2	2

LM = Linear Mode
NLM = Nonlinear Mode

R = 9.02 Ω
L = 316 mH
C = 35 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	Number of Times Circuit Breaker On
LM	46	75.8	1	1
NLM	46	107.7	1	1
LM	46	75.7	3	3
NLM	46	107.8	1	1
LM	46	74.8	24	24
NLM	46	107.5	1	1
LM	46	74.5	3	3
NLM	46	107.3	1	1
LM	46	75.8	1	1
NLM	46	107.5	1	1
NLM	46	107.5		
LM	46	74.3	3	3
NLM	46	107.3	1	1
LM	46	75	3	3
NLM	46	107.3	1	1
LM	46	74.7	5	5
NLM	46	107.2	1	1
LM	46	74.8	9	9
LM	47	81		
NLM	47	108.2	1	1
LM	47	78	48	48
NLM	47	108.2	1	1
LM	47	78.2	6	6
NLM	47	108	1	1
LM	47	78	7	7
NLM	47	108.1	1	1
LM	47	80.1	38	38
NLM	47	108.3	1	1
LM	47	78.2	8	8
NLM	47	108	1	1

LM = Linear Mode R = 9.02 Ω
NLM = Nonlinear Mode L = 316 mH
C = 35 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	47	78.3	20	20
NLM	47	108.2	1	1
LM	47	77	33	33
NLM	47	107.8	1	1
LM	47	77	2	2
NLM	47	108.1	1	1
LM	47	78.5	4	4
NLM	47	108	1	1
LM	47	78.7	22	22
NLM	47	108.2	1	1
LM	47	78	22	22
NLM	47	108	1	1
LM	47	77	5	5
NLM	47	108.3	1	1
LM	47	77.5	7	7
NLM	47	108.2	1	1
LM	47	77.8	8	8
NLM	47	108.2	1	1
LM	47	77.7	5	5
NLM	47	108.2	1	1
LM	47	77.8	8	8
NLM	47	108.2	1	1
LM	47	78.8	7	7
NLM	47	108.3	1	1
LM	47	78.7	2	2
NLM	48	108.7		
NLM	48	108.7	>100	>100
NLM	50	109.3		
NLM	50	109.3	>100	>100

LM = Linear Mode R = 9.02 Ω
NLM = Nonlinear Mode L = 316 mH
C = 35 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
NLM	60	113		
NLM	60	113	>100	>100
NLM	65	114.1		
NLM	65	114.1	>100	>100
NLM	70	115		
NLM	70	115	>100	>100
NLM	80	117		
NLM	80	117	>100	>100
NLM	90	118.5		
NLM	90	118.5	>100	>100
NLM	100	120		
NLM	100	120	>100	>100
NLM	110	121.2		
NLM	110	121.2	>100	>100
NLM	120	122.2		
NLM	120	122.2	>100	>100
NLM	130	123.2		
NLM	130	123.2	>100	>100

It is obvious that after the variac output voltage exceeds the critical voltage, it is very difficult to return transformer voltage to the linear mode.

LM = Linear Mode
NLM = Nonlinear Mode

R = 9.02 Ω
L = 316 mH
C = 55 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	45	30		
LM	45	30	>100	>100
LM	60	39		
LM	60	39	>100	>100
LM	70	45		
NLM	69.8	129	17	17
LM	70	45	1	1
NLM	70	129	8	8
LM	70	45	1	1
NLM	70	129	5	5
LM	70	45	1	1
NLM	70	129	9	9
LM	70	45	1	1
NLM	70	129	19	19
LM	70	45	1	1
NLM	70	129	1	1
LM	70	45	1	1
NLM	70	129	13	13
LM	70	45	1	1
NLM	70	129	2	2
LM	70	45	1	1
NLM	70	129	16	16
LM	70	45	1	1
NLM	70	129	15	15
LM	70	45	1	1
NLM	70	129	5	5
LM	70	45	1	1
NLM	70	129	3	3
	70	45	1	1

LM = Linear Mode
NLM = Nonlinear Mode

R = 9.02 Ω
L = 316 mH
C = 55 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker On	Off
NLM	70	129	13	13
LM	70	45	1	1
NLM	70	129	3	3
LM	70	45	1	1
NLM	70	129	1	1
LM	70	45	1	1
NLM	70	129	24	24
LM	80	51		
NLM	80	130	3	3
LM	80	51	1	1
NLM	80	130	3	3
LM	80	51	1	1
NLM	80	130	3	3
LM	80	51	1	1
NLM	80	130	1	1
LM	80	51	2	2
NLM	80	130	2	2
LM	80	51	4	4
NLM	80	130	3	3
LM	80	51	1	1
NLM	80	130	2	2
LM	80	50.7	1	1
NLM	80	130	2	2
LM	80	50.7	1	1
NLM	80	130	2	2
LM	80	50.7	3	3
NLM	80	130	1	1
LM	80	50.7	1	1
NLM	80	130	1	1

LM = Linear Mode
NLM = Nonlinear Mode

R = 9.02 Ω
L = 316 mH
C = 55 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	80	50.7	2	2
NLM	80	130	1	1
LM	80	50.7	1	1
NLM	80	130	4	4
LM	80	50.7	1	1
NLM	80	130	1	1
LM	80	50.7	2	2
NLM	80	130	1	1
LM	80	50.7	1	1
NLM	80	130	4	4
LM	80	50.7	5	5
NLM	80	130	2	2
LM	80	50.7	1	1
LM	90	56.3		
NLM	90	131	1	1
LM	90	56.2	5	5
NLM	90	131	1	1
LM	90	56	6	6
NLM	90	131	1	1
LM	90	56.2	2	2
NLM	90	131	1	1
LM	90	56.2	2	2
NLM	90	131	1	1
LM	90	56.2	2	2
NLM	90	131	1	1
LM	90	56	9	9
NLM	90	131	1	1
LM	90	56	1	1
NLM	90	131	1	1
LM	90	56.2	2	2

LM = Linear Mode R = 9.02 Ω
NLM = Nonlinear Mode L = 316 mH
C = 55 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	Number of Times Circuit Breaker On
NLM	90	131	2	2
LM	90	56.5	1	1
NLM	90	131	1	1
LM	90	56.2	1	1
NLM	90	131	1	1
LM	90	56.2	1	1
NLM	90	131	2	2
LM	90	56.5	7	7
NLM	90	131	2	2
LM	90	56.5	3	3
NLM	90	131	1	1
LM	90	56.5	15	15
NLM	90	131	1	1
LM	90	56.5	6	6
NLM	90	131	4	4
LM	90	56.1	3	3
NLM	90	131	1	1
LM	90	56.1	1	1
NLM	90	131	1	1
LM	90	56.1	4	4
LM	100	63		
NLM	100	132	1	1
LM	100	62.5	6	6
NLM	100	132	1	1
LM	100	62.8	32	32
NLM	100	132	1	1
LM	100	62.5	3	3
NLM	100	132	1	1
LM	100	62.8	16	16
NLM	100	132	1	1

LM = Linear Mode $R = 9.02\ \Omega$
NLM = Nonlinear Mode $L = 316$ mH
$C = 55\ \mu F$

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of times Circuit Breaker Off	Number of times Circuit Breaker On
LM	100	62.8	146	146
NLM	100	132	1	1
LM	100	62.8	74	74
NLM	100	132	1	1
LM	100	62.8	17	17
NLM	100	132	1	1
LM	100	62.8	15	15
NLM	100	132	1	1
LM	100	62.8	32	32
NLM	100	132	1	1
LM	100	62.8	20	20
NLM	100	132	1	1
LM	100	62.8	48	48
NLM	100	132	1	1
LM	100	62.8	8	8
NLM	100	132	1	1
LM	100	62.8	5	5
NLM	100	132	1	1
LM	100	62.8	26	26
NLM	100	132	1	1
LM	100	62.8	6	6
NLM	100	132	1	1
LM	100	62.3	66	66
NLM	100	132	1	1
LM	100	62.5	24	24
NLM	100	132	1	1
LM	100	62.5	20	20

LM = Linear Mode
NLM = Nonlinear Mode

R = 9.02 Ω
L = 316 mH
C = 55 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	110	69.8		
NLM	110	133	1	1
NLM	110	133	>300	>300
LM	120	71.5		
NLM	120	133.8	1	1
NLM	120	133.8	>300	>300
LM	130	88		
NLM	130	134.5	1	1
NLM	130	134.5	>300	>300
NLM	132	134.5	>300	>300

2. Series circuit

LM = Linear Mode R = 9.02 Ω
NLM= Nonlinear Mode L = 316 mH
C = 3.3 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	115	99		
LM	115	99	>100	>100
LM	120	100		
LM	120	100	>100	>100
LM	130	101		
LM	130	101	>100	>100
LM	135	102		
LM	135	102	>100	>100

LM = Linear Mode R = 9.02 Ω
NLM= Linear Mode L = 316 mH
C = 15 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	On
LM	94	108		
LM	94	108	>100	>100
LM	95	112		
NLM	95	150	76	76
LM	95	112	1	1
NLM	95	150	78	78
LM	95	112	1	1
NLM	95	150	58	58
LM	95	112	1	1
NLM	95	150	55	55
LM	95	112	1	1
NLM	95	150	56	56
LM	95	112	1	1
NLM	95	150	84	84
LM	95	112	1	1

(Continued)

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	Number of Times Circuit Breaker On
LM	96	112		
NLM	96	152	Without Switching	
LM	96	118	Without Switching	
NLM	96	152	Without Switching	
NLM	97	153		
NLM	97	153	>100	>100

LM = Linear Mode R = 9.02 Ω
NLM= Nonlinear Mode L = 316 mH
C = 18 μF

Mode	Variac Output Voltage (Volts-RMS)	Voltage Across Transf. (Volts-RMS)	Number of Times Circuit Breaker Off	Number of Times Circuit Breaker On
LM	115	108		
LM	115	108	>100	>100
LM	120	112		
LM	120	112	>100	>100
LM	125	115		
LM	125	115	>100	>100
LM	130	118		
LM	130	118	>100	>100
LM	135	120		
LM	135	120	>100	>100

D. Experimental Validation of the Critical Capacitance Range in Series Circuit

To verify that 1-10.5 μF capacitance range of values actually cause ferroresonance in the typical series circuit configuration, values for capacitance 3.3, 15, 10, 15, and 18, were chosen for the following experiments with I_0, m_1, and m_2 values obtained from the B-H curve:

I_0 = 0.6504065 P.U.

m_1 = 1 P.U.

m_2 = 0.0364241 P.U.

and the calculated values used

R = 9.02 (0.002537 P.U.)

L = 316 mH(0.0335069 P.U.)

The only capacitance values that caused ferroresonance were between 5 to 15 μF.

To verify that the above capacitance values will, indeed, cause ferroresonance in the typical series circuit configuration, capacitance values within and outside this range were chosen and the following experiments were conducted accordingly:

Parameters	First	Second	Third	Fourth
R (Ω)	9.02	9.02	9.02	9.02
L (mH)	316	316	316	316
C (μF)	5	10	15	18

Experiment No. 1

R = 9.02Ω
L = 316mH
Cx = 3.3μF

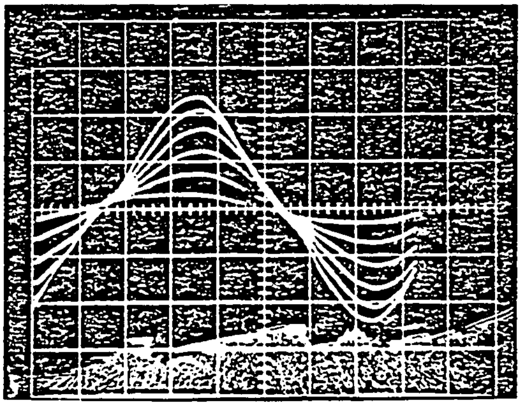


Figure 96. Voltage across transformer (V_T) (variac output between 35V–135 VRMS, voltage increments of 20 VRMS, C = 3.3 μF, scale 50 V/division)

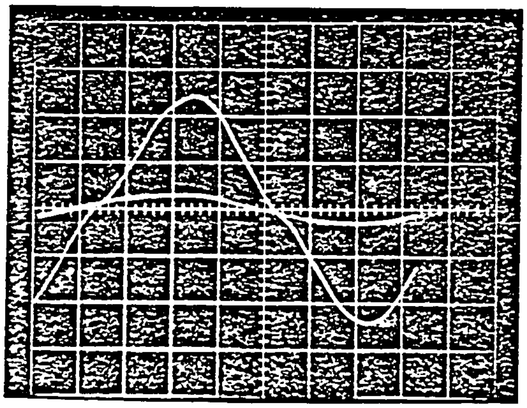
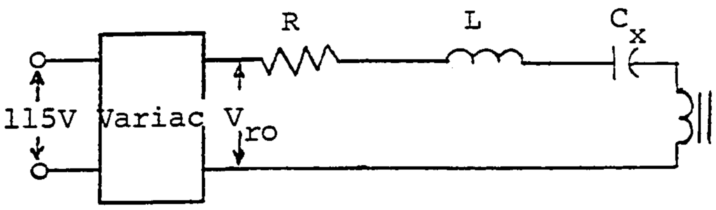


Figure 97. Voltage across transformer (V_T) (variac output 35 V and 135 V RMS, C = 3.3 μF, scale 50 V/ division)

Experiment No. 2:

R = 9.02 Ω
L = 316 mH
C_x = 5 μF



Variac Output Voltage (Volts-RMS)	Voltage Across Transformer (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Voltage Across Cap. (Volts-RMS)	Current (A-RMS)
15	5.4	6.2	24	0.01
20	11.5	9.6	37.5	0.04
25	18	12.6	47	0.07
30	24.5	15	66	0.1
35	31.5	17.8	75	0.115
40	38.2	20.3	84	0.135
45	44.5	23	92.5	0.15
F.R. Occurs				
50	61	26.2	104.0	0.175
55	67	30.1	113	0.2
60	74	35	128	0.228
65	80	40	142	0.258
70	84	45	157	0.288
75	88	50	172.5	0.315
80	90	54	185	0.34
85	92.5	59	200	0.367
90	94.5	63.2	211	0.393
95	96.5	67.6	222	0.418
100	97	71.8	235	0.442
105	97.5	75.2	243	0.462
110	99	79	260	0.482
115	100	82.5	270	0.503
120	100.1	86	280	0.52
125	100.15	89	290	0.54
130	100.2	92	300	0.558
135	100.2	95	310	0.578

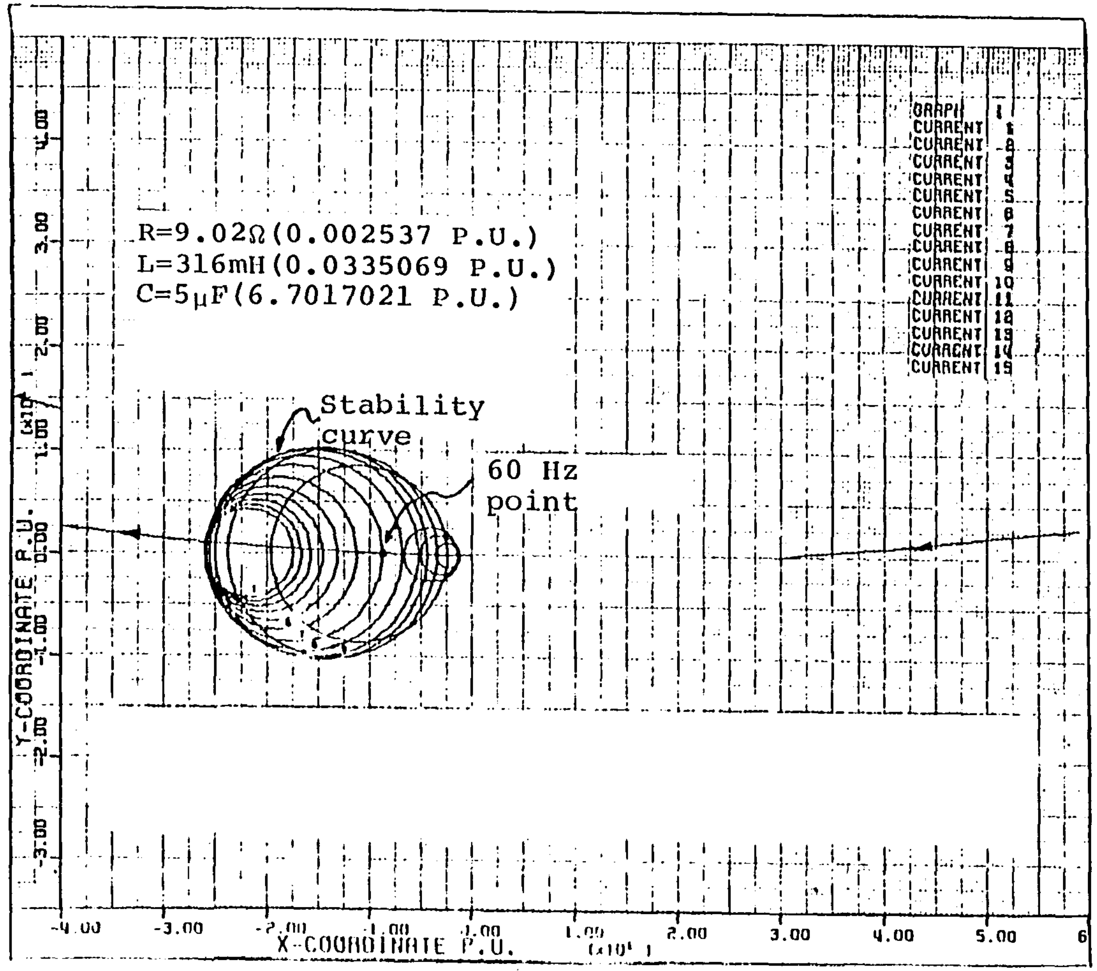


Figure 98. Stability and frequency response curves for series circuit of Experiment No. 2

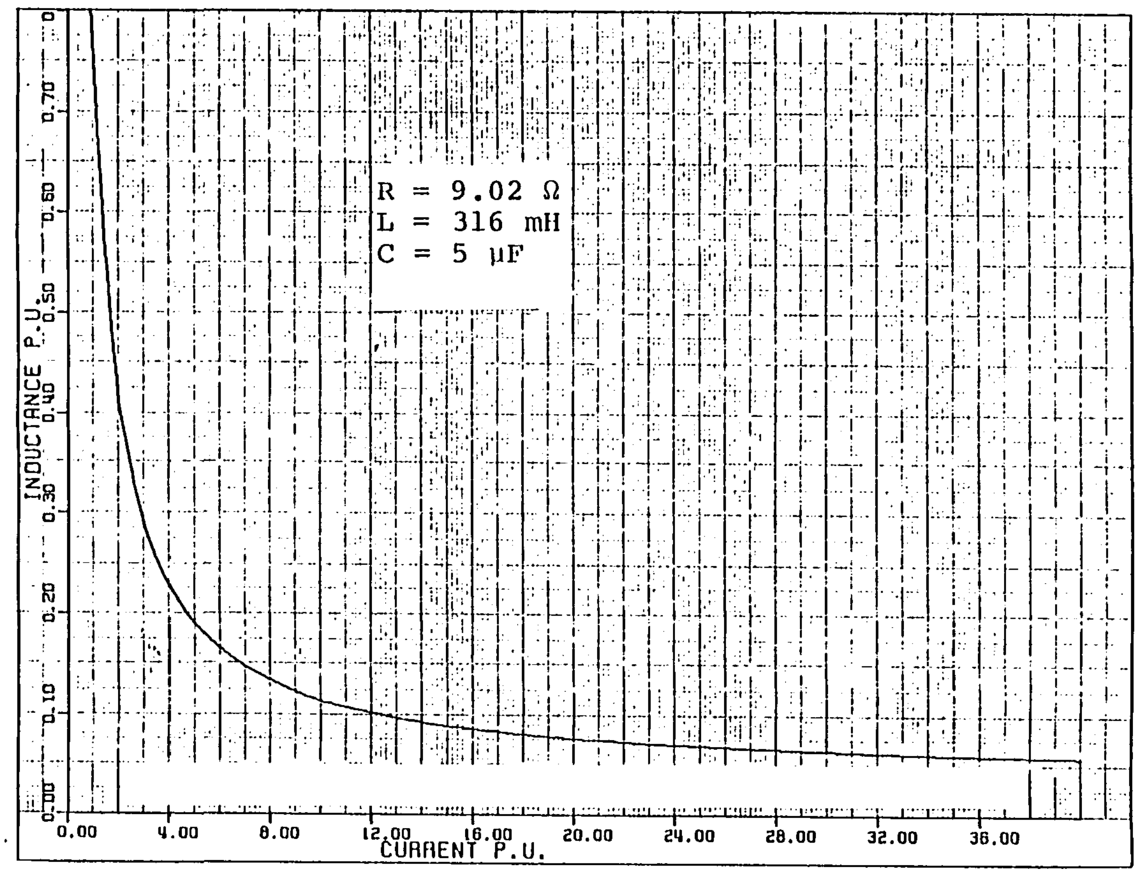


Figure 99. Inductance vs. current for series circuit of Experiment No. 2

Experiment No. 2

R = 9.02 Ω
L = 316 mH
C_x = 5 μF

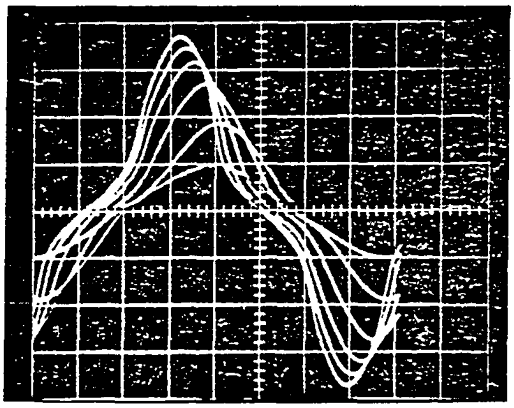


Figure 100. Voltage across transformer (V_T) (variac output between 35V-135 V RMS, voltage increments of 20 V RMS, C = 5 μF, scale 50 V/division)

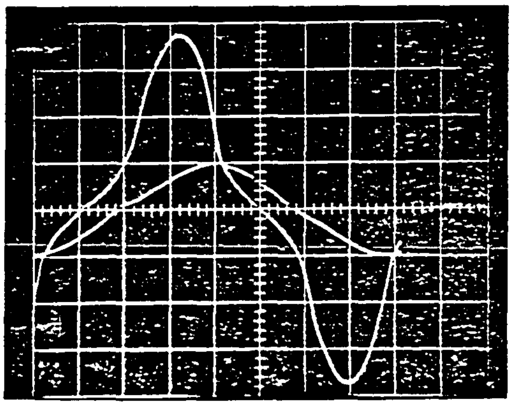


Figure 101. Voltage across transformer (V_T) (variac output 35V and 135V RMS, C = 5 μF, scale 50 V/division)

Experiment No. 3

$R = 9.02\ \Omega$
$L = 316$ mH
$C_x = 6.67\ \mu F$

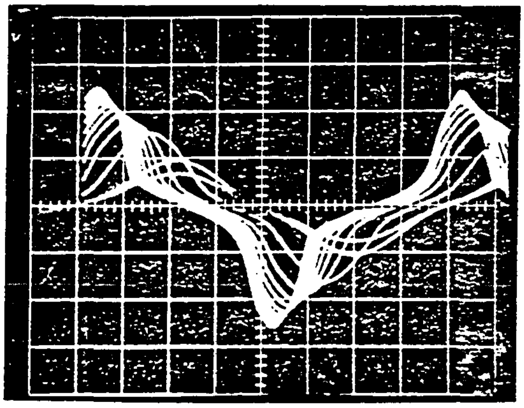


Figure 102. Voltage across transformer (V_T) (variac output between 35-135V RMS, voltage increments of 10V Rms, C = 6.67 μF, scale 100 V/division)

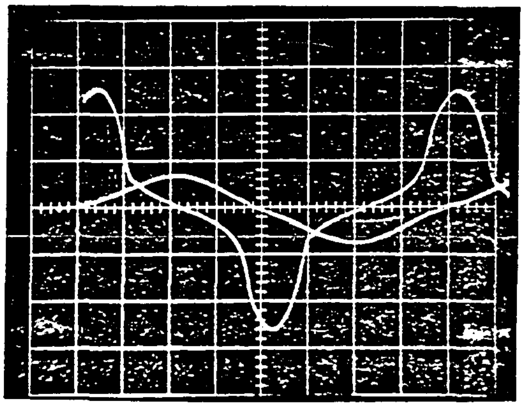


Figure 103. Voltage across transformer (V_T) (variac output 35V and 135 V RMS, C = 6.67 μF, scale 100 V/ division)

Experiment No. 4

R = 9.02 Ω
L = 316 mH
C_x= 7.5 μF

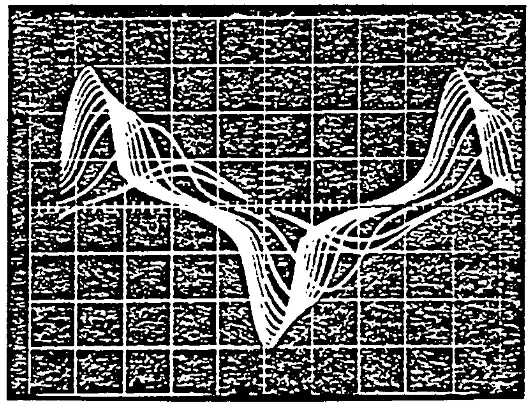


Figure 104. Voltage across transformer (V_T) (variac output between 35V–135V RMS, voltage increments of 10V RMS, C = 7.5 μF, scale 100 V/division)

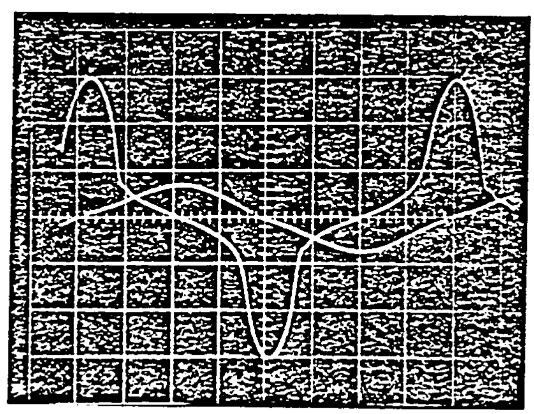


Figure 105. Voltage across transformer (V_T) (variac output 35V and 135V RMS, C = 7.5 μF, scale 100 V/ division)

<u>Experiment No. 5</u>

R = 9.02 Ω
L = 316 mH
C_x = 8 μF

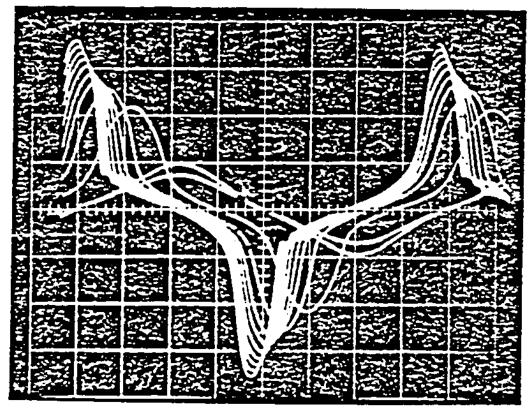


Figure 106. Voltage across transformer (V_T) (variac output between 35V-135V RMS, voltage increments of 10V RMS, C = 8 μF, scale 100 V/division)

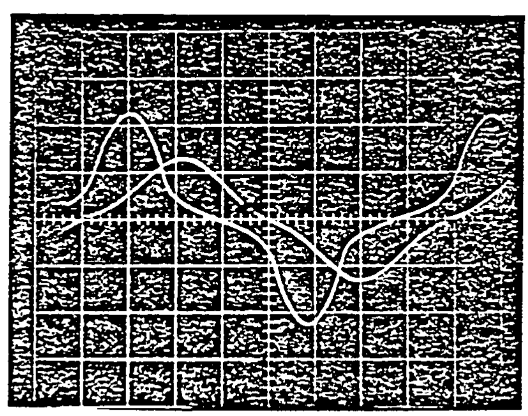


Figure 107. Voltage across transformer linear and non-linear modes (variac output at critical voltage 52.5V RMS, C = 8 μF, scale 100 V/division)

Experiment No. 6:

R = 9.02 Ω
L = 316 mH
C_x = 10 μF

Variac Output Voltage (Volts-RMS)	Voltage Across Transformer (Volts-RMS	Voltage Across Inductor (Volts-RMS)	Voltage Across Cap. (Volts-RMS)	Current (A-RMS)
15	20.2	13.2	26	0.08
20	27.3	16	31.4	0.105
25	33.4	18	35.6	0.115
30	39	20.2	39.5	0.14
35	45	22.8	44	0.151
40	60	25.8	48.8	0.18
45	66	28	63	0.2
50 F.R. occurs	74	33.6	71	0.227
55	80.5	40.2	81.1	0.27
58	120	202	347	1.258
60	120	205	351	1.2588
65	120	218	374	1.36
70	120.2	229	392	1.43
75	123	240	410	1.5
80	123.2	248	425	1.558
85	125	258	442	1.62
90	125.5	267	460	1.675
95	126	276	472	1.735
100	126.5	285	482	1.79
105	127	293	500	1.85
110	128	315	524	1.9
115	128.5	322	540	1.98
120	129	330	560	2.09
125	129.5	340	574	2.13
130	130	345	584	2.18
135	130	352	600	2.24

Experiment No. 6

$R = 9.02\ \Omega$
$L = 316$ mH
$C_x = 10\ \mu F$

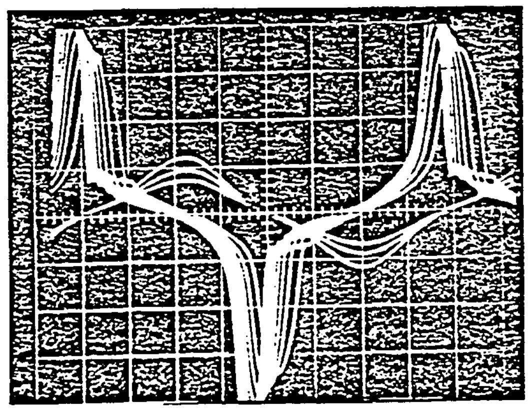


Figure 108. Voltage across transformer (V_T) (variac output between 35V-135V RMS, voltage increments of 10V RMS, C = 10 μF, scale 100 V/division)

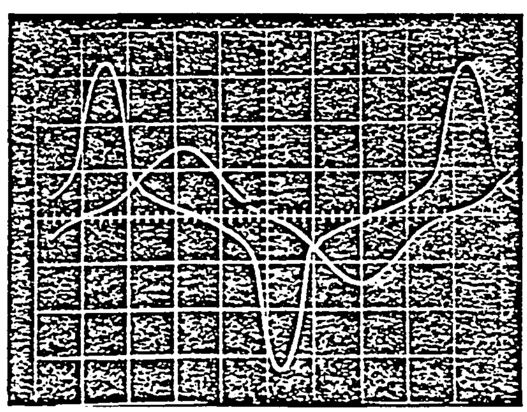


Figure 109. Voltage across transformer first and second modes (variac output at critical voltage 60.5V RMS, C = 10 μF, scale 100 V/division)

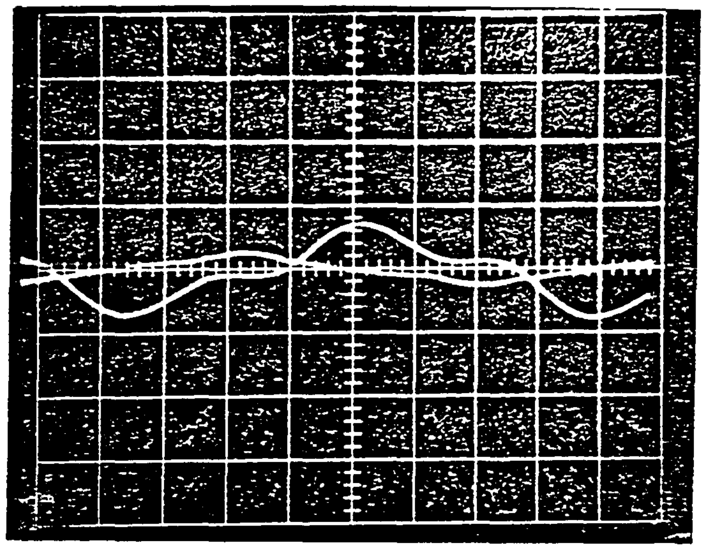


Figure 110. Transformer current linear and nonlinear modes (variac output at critical voltage 60.5 VRMS, C = 10 μF, shunt 50 MV/A, scale 0.1 V/division)

Experiment No. 6

R = 9.02Ω
L = 316mH
Cx = 10μF

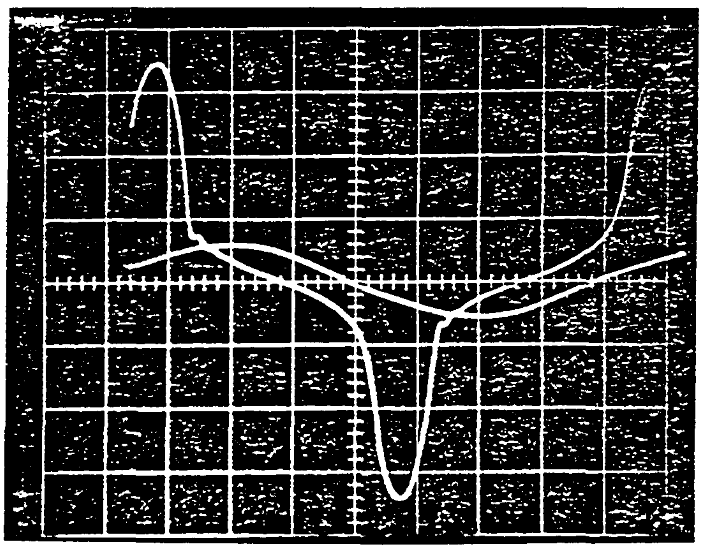


Figure 111. Voltage across transformer (V_T) just before and after loading transformer with R = 135 at F.R. (scale 100 V/division, C = 10 μF)

Experiment No. 6

$R = 9.02\ \Omega$
$L = 316$ mH
$C_x = 10\ \mu F$

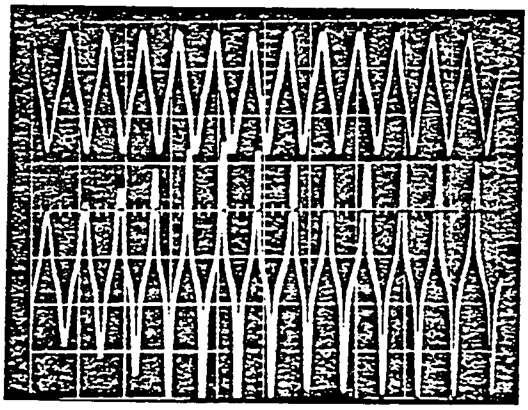


Figure 112. Voltage across transformer linear mode (upper), voltage across transformer nonlinear mode (lower) (voltage period 20 MS/division, C = 10 μF, scale 100 V/division)

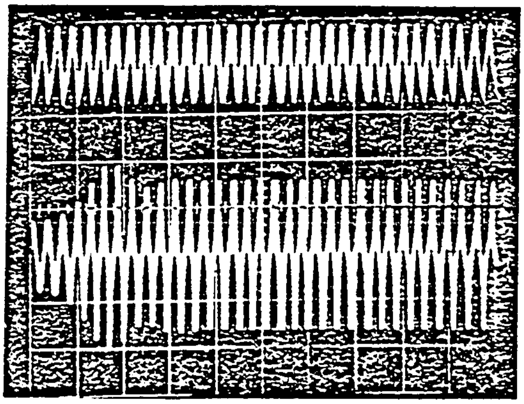


Figure 113. Voltage across transformer linear mode (upper), voltage across transformer nonlinear mode (lower) (voltage period 50 MS/division, C = 10 μF)

Experiment No. 7:

R = 9.02 Ω
L = 316 mH
C_x = 15 μF

Variac Output Voltage (Volts-RMS)	Voltage Across Transformer (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Voltage Across Cap. (Volts-RMS)	Current (A-RMS)
15	16	11.4	15.2	0.06
20	21.3	18.9	18	0.09
25	26.5	20.5	20.5	0.1005
30	31.6	17.2	23	0.115
35	37	19.5	25.7	0.13
40	42	21.5	28.2	0.14
45	47	23.7	30.7	0.16
50	61	26	33.3	0.175
55	66	28.9	36.2	0.2
60	72	32.1	40	0.22
65	76.5	36	43.9	0.241
70	80	40	48	0.268
75	84	44.4	62	0.297
80	89.9	61	70	0.336
85	93	72	78	0.39
90	99	90	96	0.493
95 F.R. occurs	106	122	127	0.673
96.5	147	830	884	5
100	154	850	1000	5.78

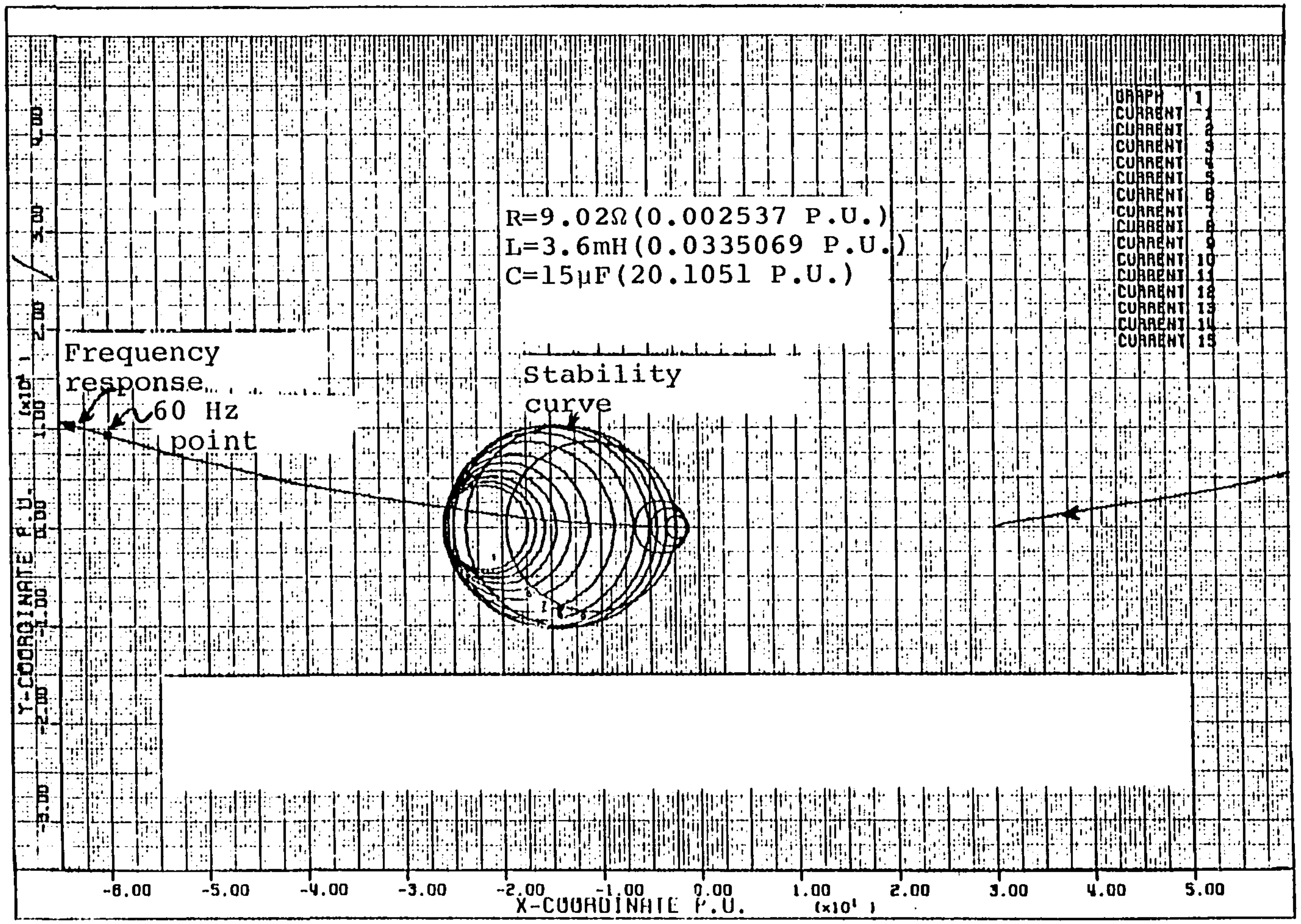


Figure 114. Stability and frequency response curves for the series circuit of Experiment No. 7

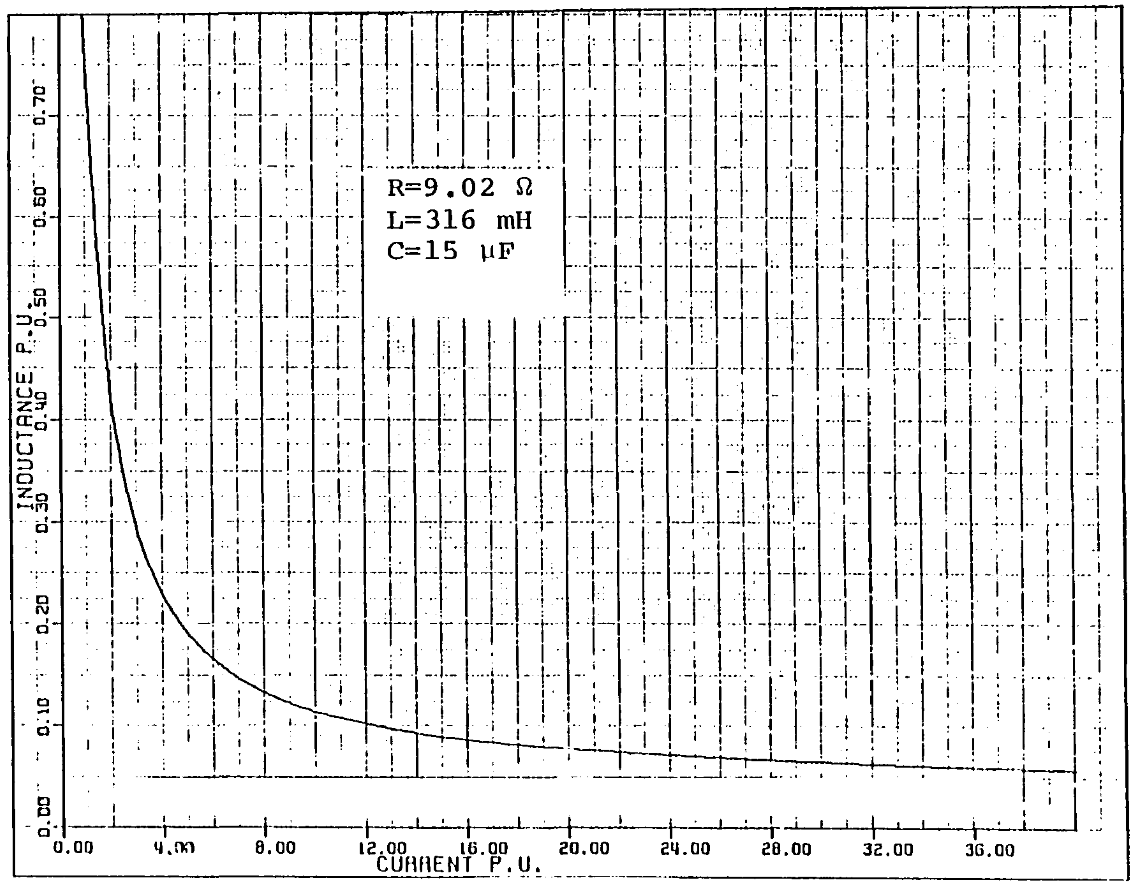


Figure 115. Inductance vs. current for series circuit of Experiment No. 7

Experiment No. 7

$R = 9.02\ \Omega$
$L = 316$ mH
$C_x = 15\ \mu F$

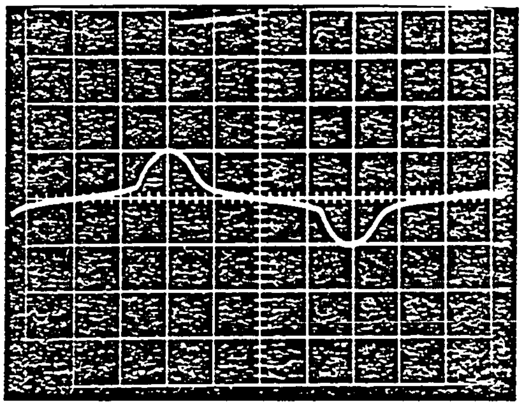


Figure 116. Voltage across transformer (V_T) (variac output 95V RMS, scale uncalibrated, calculated P-P V_T, 2.7x100 = 270 V/division, C = 15 μF)

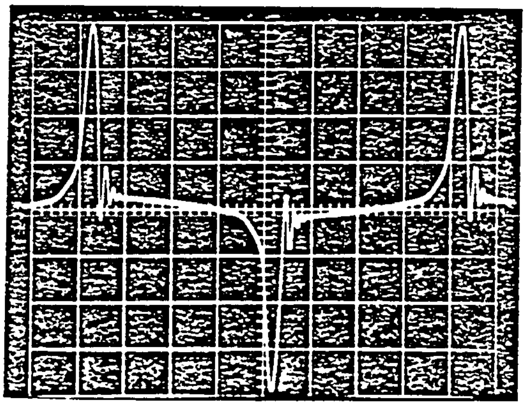


Figure 117. Voltage across transformer (V_T) (variac output 96V RMS (critical), scale uncalibrated, calculated P-P V_T, 2.7x100/division = 2133V RMS, C = 15 μF)

<u>Experiment No. 7</u>

R = 9.02 Ω
L = 316 mH
C_x = 15 μF

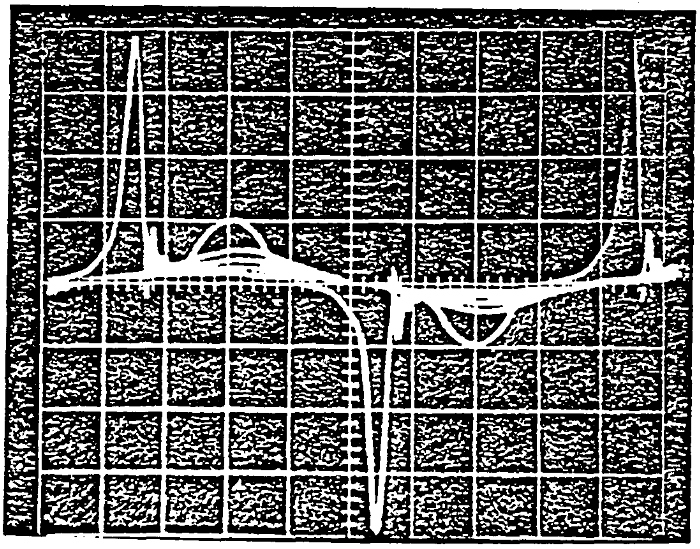


Figure 118. Voltage across transformer (V_T) (variac output between 15V-95.5V RMS, voltage increments of 20V RMS, C = 15 μF, scale uncalibrated, calculated P-P V_T, 7.9x100x2.7 = 2133V RMS)

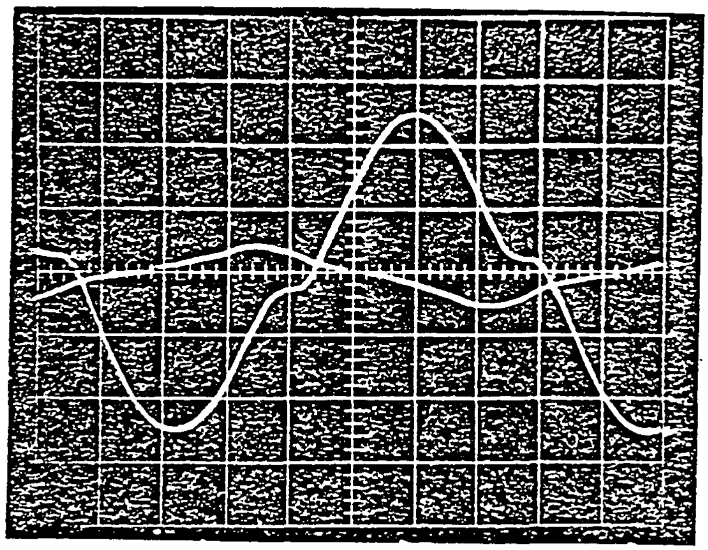


Figure 119. Transformer current linear and nonlinear modes, variac output at critical voltage 60.5 VRMS, C = 15 μF, shunt 50 MV/A, scale 0.1 V/Division)

Experiment No. 8

$R = 9.02\ \Omega$
$L = 316$ mH
$C_x = 16.67\ \mu F$

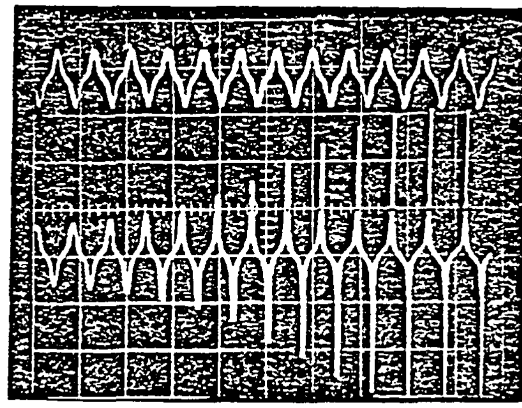


Figure 120. Voltage across transformer linear mode (upper), voltage across transformer nonlinear mode (lower) (voltage period 20 MS/division, C = 16.67 μF, scale uncalibrated)

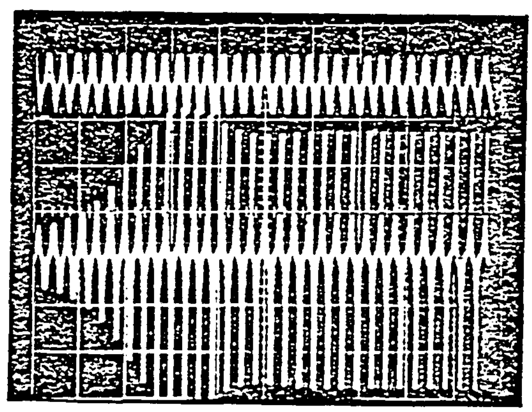


Figure 121. Voltage across transformer linear mode (upper), voltage across transformer nonlinear mode (lower) (voltage period 50 MS/division, C = 16.67 μF, scale uncalibrated)

Experiment No. 9:

$R = 9.02\ \Omega$
$L = 316$ mH
$C_x = 18\ \mu F$

Variac Output Voltage (Volts-RMS)	Voltage Across Transformer (Volts-RMS)	Voltage Across Inductor (Volts-RMS)	Voltage Across Cap. (Volts-RMS)	Current (A-RMS)
15	14	10.7	11.7	0.05
20	19.5	13	14	0.085
25	24	14.5	16	0.1
30	29	16.5	18	0.11
35	33.8	18	19.9	0.12
40	38.6	20	21.9	0.135
45	43	22	23.6	0.15
50	47.8	24	25.6	0.16
55	59.5	26	28	0.175
60	66	29	30.1	0.2
65	70	32	33	0.217
70	75	35.2	36	0.235
75	78	38.2	38.5	0.257
80	82	42	42	0.282
85	87.5	47.5	46.8	0.312
90	90	54	52	0.351
95	93	62	58.5	0.402
100	97	72	67	0.466
105	100	86	78	0.53
110	104	101.2	92	0.635
115	107	118	106	0.74
120	110	137.5	123.8	0.88
125	114	176	164	1.055
130	118	201	188	1.22
135	120	227	214	1.4

No Ferroresonance

Experiment No. 9

R = 9.02 Ω
L = 316 mH
C = 18 μF

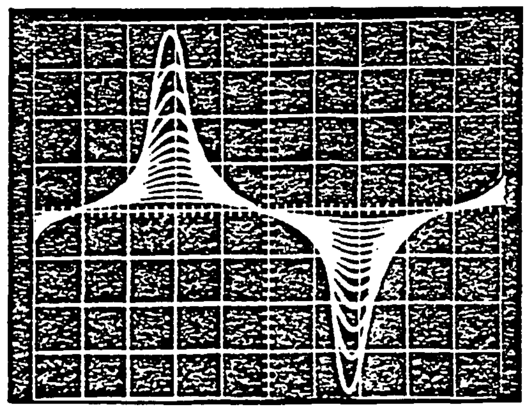


Figure 122. Voltage across transformer (V_T) (variac output between 15V-135V RMS, voltage increments of 10V RMS, C = 18 μF, scale 100 V/division)

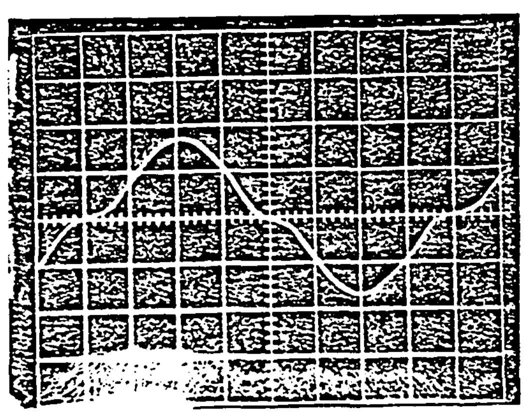


Figure 123. Transformer current (I_T) (variac output at 135V RMS, C = 25 μF, shunt 50 MV/A, scale 0.1 V/division)

X. APPENDIX B: ANALYSIS OF METHODS USED TO STUDY FERRORESONANCE

A. Piecewise Linearization Technique

For the unloaded pi circuit, the equations are:

$$RI + L\frac{dI}{dt} + \frac{d\lambda}{dt} = E_m \sin(\omega t+\phi) \qquad (10.1)$$

$$\frac{d\lambda}{dt} = \frac{1}{c}\int I_c dt = \frac{Q_c}{c} \qquad (10.2)$$

$$\lambda = b_k(I_L-a_k) + c_k \qquad (10.3)$$

$$I = I_L + I_c \qquad (10.4)$$

Since

$$\frac{d\lambda}{dI_L} = \frac{d\lambda}{dt} \cdot \frac{dt}{dI_L} = b_k$$

$$\therefore \frac{d\lambda}{dt} \cdot \frac{dt}{dI_L} = b_k = L_k \text{ and } \frac{d\lambda}{dt} = L_k \frac{dI_L}{dt} \qquad (10.5)$$

and

$$\frac{d\lambda}{dt} = \frac{Q_c}{c}$$

$$\therefore L_k \frac{dI_L}{dt} = \frac{Q_c}{c} \text{ and } \frac{dI_L}{dt} = \frac{Q_c}{L_k c}$$

since

$$I = I_L + I_c \qquad \text{and} \qquad c\frac{d\dot{\lambda}}{dt} = \int I_c dt = Q_c$$

$$I = I_L + \frac{dQ_c}{dt} \qquad cL_k \frac{dI_L}{dt} = \int I_c dt = Q_c$$

$$cL_k \frac{d^2 I_L}{dt^2} = I_c = \frac{dQ_c}{dt}$$

$$\therefore I = I_L + cL_k \frac{d^2 I_L}{dt^2} \qquad cL_k \frac{d^3 I_L}{dt^3} = \frac{dI_c}{dt}$$

$$RI + L\frac{dI}{dt} + L_k \frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

$$RI + L\frac{dI}{dt} + \frac{Q_c}{c} = E_m \sin(\omega t+\phi)$$

$$CRI + LC\frac{dI}{dt} + Q_c = E_m C \sin(\omega t+\phi)$$

$$Q_c = E_m C \sin(\omega t+\phi) - CRI - LC\frac{dI}{dt} \tag{10.6}$$

$$RI + L\frac{dI}{dt} + L_k \frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

$$R(I_L+I_c) + L\frac{d}{dt}(I_L+I_c) + L_k \frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

$$RI_L+RI_c + L\frac{dI_L}{dt} + L\frac{dI_c}{dt} + L_k \frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

$$RI_L+RI_c + (L+L_k)\frac{dI_L}{dt} + L\frac{dI_c}{dt} = E_m \sin(\omega t+\phi)$$

$$RI_L+R(CL_k \frac{d^2 I_L}{dt^2}) + (L+L_k)\frac{dI_L}{dt} + L(CL_k)\frac{d^3 I_L}{dt^3} = E_m \sin(\omega t+\phi)$$

$$LCL_k \frac{d^3 I_L}{dt^3} + RCL_k \frac{d^2 I_L}{dt^2} + (L+L_k)\frac{dI_L}{dt} = E_m \sin(\omega t+\phi)$$

or

$$\frac{d^3 I_L}{dt^3} + \frac{RCL_k}{LCL_k}\frac{d^2 I_L}{dt^2} + \left(\frac{L+L_k}{LCL_k}\right)\frac{dI_L}{dt} + \frac{RI_L}{LCL_k} = \frac{E_m}{LCL_k}\sin(\omega t+\phi) \quad (10.7)$$

characteristic equation:

$$m^3 + \frac{R}{L}m^2 + \frac{L+L_k}{LCL_k}m + \frac{R}{LCL_k} = 0$$

Roots may be found by the numerical analysis technique called the secant method. However, since the last term is less significant compared to the remaining terms, we will neglect it to simplify the solution.

$$m^3 + \frac{R}{L}m^2 + \frac{L+L_k}{LCL_k}m = 0$$

$$m\left(m^2 + \frac{R}{L}m + \frac{L+L_k}{LCL_k}\right) = 0$$

$$m_1 = 0$$

$$m_{2,3} = -\frac{R}{2L} \mp \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - 4\left(\frac{L+L_k}{LCL_k}\right)}$$

$$= -\frac{R}{2L} \mp \sqrt{\frac{R^2}{4L^2} - \frac{4(L+L_k)}{4(LCL_k)}}$$

where $L_{eq} = \frac{LL_k}{L+L_k}$

and

$$m_{2,3} = -\frac{R}{2L} \mp \sqrt{\frac{R^2}{4L^2} - \frac{1}{L_{eq}C}}$$

$$= -\frac{R}{2L} \mp j\sqrt{\frac{1}{L_{eq}C} - \frac{R^2}{4L^2}}$$

$$m_{2,3} = -\frac{R}{2L} \mp j\nu$$

$$\therefore\ I_{compl.} = Ae^{ot} + Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2)$$

or

$$I_{compl.} = A + Be^{-\frac{R}{2L}}\sin(\nu t+\theta_2)$$

$$I_{particular_1} = \frac{\frac{E_m}{LCL_k} \cdot e^{j\omega(t+\frac{\phi}{\omega})}}{(+j\omega)^3 + \frac{R}{L}(j\omega)^2 + \frac{L+L_k}{LL_kC}(j\omega)}$$

$$I_{p_1} = \frac{\frac{E_m}{LCL_k}}{-j\omega^3 - \frac{R}{L}\omega^2 + j(\frac{L+L_k}{LL_kC}\omega)} e^{j(\omega t+\phi)}$$

$$= \frac{E_m \qquad e^{j(\omega t+\phi)}}{-j\omega^3 LCL_k - CL_kR\omega^2 + j(L+L_k)\omega}$$

$$I_{p_1} = \frac{E_m \qquad e^{j(\omega t+\phi)}}{-CL_kR\omega^2 + j[(L+L_k)\omega - LCL_k\omega^3]}$$

$$= \frac{\frac{E_m}{\omega^2} \qquad e^{j(\omega t+\phi)}}{-CL_kR + j[\frac{(L+L_k)}{\omega} - LL_kC\omega]}$$

$$= \frac{\frac{E_m}{LL_k\omega^2} \qquad e^{j(\omega t+\phi)}}{-\frac{CR}{L} + j[(\frac{L+L_k}{LL_k\omega}) - C\omega]}$$

$$= \frac{E_m}{LL_k\omega^2}\left[\frac{1 \qquad e^{j(\omega t+\phi)}}{-\frac{CR}{L} + j(\frac{1}{L_{eq}\omega} - C\omega)}\right]$$

$$= \frac{E_m}{LL_k\omega^2}\left[\frac{1 \qquad e^{j(\omega t+\phi)}}{\sqrt{(\frac{CR}{L})^2 + (\frac{1}{L_{eq}\omega} - C\omega)^2} \angle -\tan^{-1}\frac{\frac{1}{L_{eq}\omega} - C\omega}{\frac{CR}{L}}}\right]$$

$$= \frac{E_m}{LL_k\omega^2}\left[\frac{1 \angle \cdot\tan^{-1}\frac{\frac{1}{L_{eq}\omega} - C\omega}{\frac{CR}{L}}}{\sqrt{(\frac{CR}{L})^2 + (\frac{1}{L_{eq}\omega} - C\ )^2}}\right]e^{+j(\omega t+\phi)}$$

$$= \frac{E_m}{LL_k\omega^2}\frac{1}{Z}e^{j\theta_1} \cdot e^{j(\omega t+\phi)}$$

$$I_{p_1} = \frac{E_m}{LL_k\omega^2 Z}e^{j(\omega t+\phi+\theta_1)}$$

Since the source function is sinusoidal:

$$I_p = \frac{E_m}{LL_k\omega^2 Z}\sin(\omega t+\phi+\theta_1) = \frac{E_m}{XX_k Z}\sin(\omega t+\phi+\theta_1) \qquad (10.8)$$

where

$$\theta_1 = \tan^{-1}\frac{\frac{1}{L_{eq}\omega} - C\omega}{\frac{CR}{L}}$$

$$L_{eq} = \frac{LL_k}{L+L_k}$$

$$Z = \sqrt{\left(\frac{CR}{L}\right)^2 + \left(\frac{1}{L_{eq}\omega} - C\omega\right)^2} \qquad (10.9)$$

Solution:

$$I = I_c+I_p$$

$$I_k = A+Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) + \frac{E_m}{LL_k\omega^2 Z}\sin(\omega t+\phi+\theta_1) \qquad (10.10)$$

1. Evaluating A, B, and θ_2 for the first segment

We need three relations:

a) Set $t = 0$, $I = I_0$

$$\therefore I_0 = A + B\sin\theta_2 + \frac{E_m}{LL_k\omega^2 Z}\sin(\phi+\theta_1) \qquad (10.11)$$

since

$$Q_c = CL_k\frac{dI_L}{dt}$$

$$\therefore Q_c = CL_k[0+(-\frac{R}{2L})(B)\sin(\nu t+\theta_2)e^{-\frac{R}{2L}t}+Be^{-\frac{R}{2L}t}\nu\cos(\nu t+\theta_2)$$

$$+ \frac{E_m\omega}{LL_k\omega^2 Z}\cos(\omega t+\phi+\theta_1)]$$

$$Q_c = -\frac{CL_kRB}{2L}\sin(\nu t+\theta_2)e^{-\frac{R}{2L}t}+CL_k\nu Be^{-\frac{R}{2L}t}\cos(\nu t+\theta_2)$$

$$+ \frac{CE_m}{L\omega Z}\cos(\omega t+\phi+\theta_1)]$$

$$\therefore Q_o = -\frac{CL_kRB}{2L}\sin(\theta_2) + CL_k\nu B\cos\theta_2$$

$$+ \frac{CE_m}{L\omega Z}\cos(\phi+\theta_1) \qquad (10.12)$$

The third relation is found by substituting the solution in the differential equation:

$$\frac{d^3I_L}{dt^3} + \frac{R}{L}\frac{d^2I_L}{dt^2} + \frac{L+L_k}{LCL_k}\frac{dI_L}{dt} = \frac{E_m}{LCL_k}\sin(\omega t+\phi)$$

where

$$I_L = A + Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) + \frac{E_m}{LL_k\omega^2 Z}\sin(\omega t+\phi+\theta_1)$$

$$\frac{dI_L}{dt} = 0 - \frac{R}{2L}Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) + Be^{-\frac{R}{2L}t}\nu\cos(\nu t+\theta_2)$$

$$+ \frac{E_m\omega}{LL_k\omega^2 Z}\cos(\omega t+\phi+\theta_1)$$

$$\frac{d^2I_L}{dt^2} = +\frac{R^2}{4L^2} Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) - \frac{R}{2L}B\nu e^{-\frac{R}{2L}t}\cos(\nu t+\theta_2)$$

$$-\frac{R}{2L}Be^{-\frac{R}{2L}t}\nu\cos(\nu t+\theta_2) - B\nu^2 e^{-\frac{R}{2L}t}\sin(\nu t+\theta_2)$$

$$-\frac{E_m\omega^2}{LL_k\omega^2 Z}\sin(\omega t+\phi+\theta_1)$$

$$\frac{d^3I_L}{dt^3} = -\frac{R^3}{8L^3}Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) + \frac{3R^2}{4L^2}Be^{-\frac{R}{2L}t}\cos(\nu t+\theta_2)$$

$$+\frac{3R}{2L}B\nu^2 e^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) - B\nu^3 e^{-\frac{R}{2L}t}\cos(\nu t+\theta_2)$$

$$-\frac{E_m\omega^3}{LL_k\omega^2 Z}\cos(\omega t+\phi+\theta_1)$$

Substituting solution in the third order differential equation:

$$-\frac{R^3}{8L^3}Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) + \frac{3R^2}{4L^2}Be^{-\frac{R}{2L}t}\cos(\nu t+\theta_2)$$

$$+\frac{3R}{2L}B\nu^2 e^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) - B\nu^3 e^{-\frac{R}{2L}t}\cos(\nu t+\theta_2)$$

$$-\frac{E_m\omega^3}{LL_k\omega^2 Z}\cos(\omega t+\phi+\theta_1) + \frac{R^3}{4L^3}Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2)$$

$$-\frac{R^2}{L^2}B\nu e^{-\frac{R}{2L}t}\cos(\nu t+\theta_2) - \frac{R}{L}B\nu^2 e^{-\frac{R}{2L}}\sin(\nu t+\theta_2)$$

$$-\frac{E_m\omega^2 R}{L^2L_k\omega^2 Z}\sin(\nu t+\phi+\theta_1) + \left(\frac{L+L_k}{LL_kC}\right)\frac{R}{2L}Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2)$$

$$+\left(\frac{L+L_k}{LL_kC}\right)Be^{-\frac{R}{2L}t}\nu\cos(\nu t+\theta_2) + \left(\frac{L+L_k}{LL_kC}\right)\frac{E_m\omega}{LL_k\omega^2 Z}\cos(\omega t+\phi+\theta_1)$$

$$= \frac{E_m}{LL_kC}\sin(\omega t+\phi)$$

<u>This equation is evaluated at t=0</u>:

$$\frac{R^3}{8L^3}B\sin(\theta_2) - \frac{R^2}{4L^2}B\nu\cos(\theta_2) + \frac{R}{2L}B\nu^2\sin\theta_2$$

$$-B\lambda^3\cos\theta_2 - \frac{\omega E_m}{LL_kZ}\cos(\phi+\theta_1) - \frac{E_mR}{ZL^2L_k}\sin(\phi+\theta_1)$$

$$-\frac{1}{L_{eq}C}\frac{RB}{2L}\sin(\theta_2) + \frac{1}{L_{eq}C}B\nu\cos\theta_2$$

$$+\frac{1}{L_{eq}C}\frac{E_m}{LL_k\omega Z}\cos(\phi+\theta_1) = \frac{E_m}{LL_kC}\sin(\omega t+\phi)$$

$$\sin\theta_2\left[\frac{R^3B}{8L^3} + \frac{R}{2L}B\nu^2 - \frac{1}{L_{eq}C}\frac{RB}{2L}\right] + \cos\theta_2\left[-B\nu^3 - \frac{R^2B\nu}{4L^2} + \frac{B\nu}{L_{eq}C}\right]$$

$$+\cos(\phi+\theta_1)\left[-\frac{E_m\omega}{LL_kZ} + \frac{E_m}{L_{eq}CLL_k\omega Z}\right] - \frac{E_mR}{ZL^2L_k}\sin(\phi+\theta_1)$$

$$= \frac{E_m}{LL_kC} \sin(\phi) \tag{10.13}$$

$$B \sin \theta_2 \left[\frac{R^3}{8L^3} + \frac{R\nu^2}{2L} - \frac{R}{2L_{eq}CL}\right] + B \cos \theta_2\left[- \nu^3 - \frac{R^2\nu}{4L^2} + \frac{\nu}{L_{eq}C}\right]$$

$$+ \cos(\phi+\theta_1) \left[\frac{-E_m\omega}{LL_kZ} + \frac{E_m}{L_{eq}CLL_k\omega Z}\right] - \frac{E_mR}{ZL_kL^2} \sin(\phi+\theta_1)$$

$$= \frac{E_m}{LL_kC} \sin \phi$$

$$B \sin \theta_2\left[\frac{R^3}{8L^3} + \frac{R\nu^2}{2L} - \frac{R}{2L_{eq}CL}\right] + B \cos \theta_2 \left[-\nu^3 + \frac{\nu}{L_{eq}C} - \frac{R^2}{4L^2} B\nu\right]$$

$$+ \cos(\phi+\theta_1) \left[\frac{-E_m\omega^2L_{eq}C+E_m}{LL_kZL_{eq}C\omega}\right] - \frac{E_mR}{ZL_kL^2} \sin(\phi+\theta_1)$$

$$= \frac{E_m}{LL_kC} \sin \phi$$

$$B\left[\sin(\theta_2 \frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}) - \cos \theta_2 (\frac{R^2}{4L^2}\nu+\nu^3-\frac{\nu}{L_{eq}C})\right]$$

$$= \frac{E_m}{LL_kC} \sin\theta - \cos(\phi+\theta_1) \left[\frac{-E_m\omega^2CL_{eq}+E_m}{LL_kZL_{eq}C\omega}\right] + \frac{E_mR}{L^2L_kZ} \sin(\phi+\theta_1)$$

$$\therefore B = \frac{\frac{E_m}{LL_kC} \sin\phi - \cos(\phi+\theta_1)\left[\frac{-E_m\omega^2CL_{eq}+E_m}{LL_kZL_{eq}C\omega}\right] + \frac{E_mR}{L^2L_kZ} \sin(\phi+\theta_1)}{\sin \theta_2 \left(\frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}\right) - \cos \theta_2\left(\frac{R^2}{4L^2} + \nu^3 - \frac{\nu}{L_{eq}C}\right)}$$

From equation 10.12:

$$Q_0 = \frac{-CL_kR}{2L} \sin\ \theta_2[B] + CL_k\ \nu\cos\ \theta_2[B] + \frac{CE_m}{L\omega Z} \cos(\phi+\theta_1)$$

$$Q_0 = \frac{-CL_kR}{2L} \sin\ \theta_2\ x$$

$$x\ [\frac{\frac{E_m}{LL_kC} \sin\phi - \cos(\phi+\theta_1)\ (\frac{-E_m\omega^2CL_{eq}+E_m}{LL_kZL_{eq}C\omega}) + \frac{E_mR}{L^2L_kZ} \sin(\phi+\theta_1)}{\sin\ \theta_2(\frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}) - \cos\ \theta_2(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C})}]$$

$$+ CL_k\ \nu\cos\ \theta_2\ x$$

$$x\ [\frac{\frac{E_m}{LL_kC} \sin\phi - \cos(\phi+\theta_1)\ (\frac{-E_m\omega^2CL_{eq}+E_m}{LL_kZL_{eq}C\omega}) + \frac{E_mR}{L^2L_kZ} \sin(\phi+\theta_1)}{\sin\ \theta_2(\frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}) - \cos\ \theta_2\ (\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C})}]$$

$$+ \frac{CE_m}{L\omega Z} \cos(\phi+\theta_1)$$

or

$$Q_0 2L[\sin\ \theta_2\ (\frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}) - \cos\theta_2(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C})]$$

$$= \sin\ \theta_2[R\frac{-E_m}{L} \sin\phi + \cos(\phi+\theta_1)\{\frac{-RE_m\omega^2CL_{eq}+E_mR}{LZL_{eq}\omega}\}$$

$$- \frac{E_m R^2 C}{L^2 Z} \sin(\phi+\theta_1)] + \cos\theta_2 \; [2\nu E_m \sin\phi - \cos(\phi+\theta_1)$$

$$x\{\frac{-2\nu E_m \omega^2 C L_{eq} + 2E_m \nu}{Z L_{eq}\omega}\} + \frac{2E_m RC\nu}{LZ} \sin(\phi+\theta_1)]$$

$$+ \frac{2CE_m}{\omega Z} \cos(\phi+\theta_1) \; [\sin\theta_2 (\frac{R^3}{8L^3} - \frac{\nu^2 R}{2L} - \frac{R}{2L_{eq}CL})$$

$$- \cos\theta_2 (\frac{R^3}{4L^2}\nu + \nu^3 - \frac{\nu}{L_{eq}C})]$$

Divide by $\cos\theta_2$:

$$(Q_0)(2L)[\tan\theta_2 (\frac{R^3}{8L^3} - \frac{\nu^2 R}{2L} - \frac{R}{2L_{eq}CL}) - (\frac{R^2}{4L^2}\nu + \nu^3 - \frac{\nu}{L_{eq}C})]$$

$$= \tan\theta_2 [\frac{-RE_m}{L} \sin\phi + \cos(\phi+\theta_1)\{\frac{-RE_m\omega^2 C L_{eq} + E_m R}{LZL_{eq}\omega}\}$$

$$- \frac{E_m R^2 C}{L^2 Z} \sin(\phi+\theta_1)] + [\frac{2\nu E_m}{L} \sin\phi - \cos(\phi+\theta_1)$$

$$\{\frac{2\nu E_m \omega^2 C L_{eq} + 2E_m \nu}{Z L_{eq}\omega}\} + \frac{2E_m RC\nu}{LZ} \sin(\phi+\theta_1)] + \frac{2CE_m}{\omega Z}\cos(\phi+\theta_1)$$

$$x \; [\tan\theta_2 (\frac{R^3}{8L^3} - \frac{\nu 2 R}{2L} - \frac{R}{2L_{eq}CL}) - (\frac{R^2}{4L^2}\nu + \nu^3 - \frac{\nu}{L_{eq}C})]$$

$$2LQ_0\left(\frac{R^3}{8L^3} - \frac{\nu^2 R}{2L} - \frac{R}{2L_{eq}CL}\right)\tan\theta_2 - 2LQ_0\left(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C}\right)$$

$$= \tan\theta_2\left[\frac{-RE_m}{L}\sin\phi - \frac{(E_mR-RE_m\omega^2CL_{eq})}{LZL_{eq}\omega}\cos(\phi+\theta_1)\right.$$

$$\left. - \frac{E_mR^2C}{L^2Z}\sin(\phi+\theta_1)\right] + 2\nu E_m \sin\phi - \left(\frac{E_m-\nu E_m\omega^2CL_{eq}}{ZL_{eq}\omega}\right)\cos(\phi+\theta_1)$$

$$+ \frac{2E_mRC\nu}{LZ}\sin(\phi+\theta_1) + \frac{2CE_m}{L\omega Z}\left(\frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}\right)$$

$$\times \cos(\phi+\theta_1)\tan\theta_2 - \frac{2CE_m}{\omega Z}\left(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{2L_{eq}C}\right)\cos(\phi+\theta_1)$$

or

$$2LQ_0\left(\frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}\right)\tan\theta_2$$

$$- \left[\frac{-RE_m}{L}\sin\theta + \frac{(ER_m-RE_m\omega^2CL_{eq})}{LZL_{eq}\omega}\cos(\phi+\theta_1)\right.$$

$$\left. - \frac{E_mR^2C}{L^2Z}\sin(\phi+\theta_1)\right]\tan\theta_2 - \frac{2CE_m}{\omega Z}\left(\frac{R^3}{8L^3} - \frac{\nu^2R}{2L} - \frac{R}{2L_{eq}CL}\right)$$

$$\times \cos(\phi+\theta_1)\tan\theta_2$$

$$= 2LQ_0\left(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C}\right) + 2\nu E_m\sin\phi - \left(\frac{2\nu E_m-2\nu E_m\omega^2CL_{eq}}{ZL_{eq}\omega}\right)$$

$$\times \cos(\phi+\theta_1) + \frac{2E_m RC\nu}{LZ}\sin(\phi+\theta_1) + \frac{2CE_m}{L\omega Z}\left(\frac{R^2}{4L^2}\nu+\nu^3 + \frac{\nu}{L_{eq}C}\right)$$

$$\times \cos(\phi+\theta_1)$$

$$\therefore \tan\theta_2 = \frac{\begin{array}{l} 2LQ_0\left(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C}\right) + \left[\frac{2\nu E_m}{L}\sin\phi - \right. \\ \left. -\left(\frac{2\nu E_m - 2\nu E_m\omega^2 CL_{eq}}{ZL_{eq}}\right)\cos(\phi+\theta_1) + \frac{2E_m RC\nu}{LZ}\sin(\phi+\theta_1)\right] \\ + \frac{2CE_m}{\omega Z}\left(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C}\right)\cos(\phi+\theta_1) \end{array}}{\begin{array}{l} 2LQ_0\left(\frac{R^3}{8L^3} - \frac{\nu^2 R}{2L} - \frac{R}{2L_{eq}CL}\right) - \left[\frac{-RE_m}{L}\sin\theta \right. \\ \left. + \frac{(E_m R - RE_m\omega^2 CL_{eq})}{LZL_{eq}\omega}\cos(\phi+\theta_1) - \frac{E_m R^2 C}{L^2 Z}\sin(\phi+\theta_1)\right] \\ - \left[\frac{2CE_m}{\omega Z}\left(\frac{R^3}{8L^3} - \frac{\nu^2 R}{2L} - \frac{R}{2L_{eq}CL}\right)\cos(\phi+\theta_1)\right] \end{array}}$$

$\therefore \theta_2$ is known as well as A and B.

Then, we substitute these values in the solution and we get

$$I_L = A + Be^{-\frac{R}{2L}t} \sin(\nu t+\theta_2) + \frac{E_m}{LL_k\omega^2 Z} \sin(\omega t+\phi+\theta_1)$$

Solution for the first piece of magnetization curve, namely $b_1 = L_1 = 1$ and

$$I_0 = 0$$

$$Q_0 = 0$$

$$I_L = A + Be^{-\frac{R}{2L}t} \sin(\nu+\theta_2) + \frac{E_m}{LL_1\omega^2 Z} \sin(\omega t+\phi+\theta_1)$$

where

a) $L_{eq} = \dfrac{L_1 L}{L_1+L}$

b) $I_0 = 0 = A+B \sin \theta_2 + \dfrac{E_m}{LL_1\omega^2 Z} \sin(\phi+\theta_1)$

c) $Q_0 = 0 = -\dfrac{CL_1 RB}{2L} \sin \theta_2 + CL_1 \nu B \cos \theta_2$

$$+ \frac{CE_m}{L\omega Z} \cos(\theta+\phi_1)$$

$$0+ \frac{2\nu E_m}{L} \sin\phi - \left(\frac{2\nu E_m - 2\nu E_m \omega^2 C L_{eq}}{Z L_{eq} \omega}\right) \cos(\phi+\theta_1) \ +$$

d)

$$\tan\theta_2 = \frac{+\frac{2E_m RC\nu}{LZ} \sin(\phi+\theta_1) + \frac{2CE_m}{\omega Z}\left(\frac{R^2}{4L^2}\nu+\nu^3 - \frac{\nu}{L_{eq}C}\right)\cos(\phi+\theta_1)}{0 - \left[\frac{-RE_m}{L}\sin\theta + \left(\frac{E_m R - RE_m \omega^2 C L_{eq}}{LZL_{eq}\omega}\right)\cos(\phi+\theta_1) - \frac{E_m R^2 C}{L^2 Z}\sin(\phi+\theta_1)\right] - \left[\frac{2CE_m}{L\omega Z}\left(\frac{R^3}{8L^3} - \frac{\nu^2 R}{2L} - \frac{2}{2L_{eq}CL}\right) \times \cos(\phi+\theta_1)\right]}$$

e)

$$B = \frac{\frac{E_m}{LL_1 R}\sin \ - \cos(\phi+\theta_1)\left(\frac{E_m - E_m\omega^2 C L_{eq}}{LL_1 Z L_{eq} C\omega}\right) + \frac{E_m R}{L^2 L_k Z}\sin(\phi+\theta_1)}{\sin\theta_2\left(\frac{R^3}{8L^3} - \frac{\nu^2 R}{2L} - \frac{R}{2L_{eq}L}\right) - \cos\theta_2\left(\frac{R^2}{4L^2} + \nu^3 - \frac{\nu}{L_{eq}C}\right)}$$

A guess is made as to the time t' the current function crosses the limit $I = A_2$, and the value of t in the above equation corresponding to that value is found by solving for t by using Newton-Raphson method in the following manner:

$$\dot{f}(t) = \frac{df(t)}{dt} = \frac{dI_L}{dt} = 0 + B\left(\frac{-R}{2L}\right)e^{-\frac{R}{2L}t}\sin(\nu t+\theta_2)$$

$$+ Be^{-\frac{R}{2L}t}\nu\cos(\nu t+\theta_2) + \frac{E_m\omega}{LL_1\omega^2 Z}\cos(\omega t+\theta_1+\phi)$$

$$\Delta t = \frac{a_2-f(t')}{\dot{f}(t')} \quad \text{is known and} \quad t = t'+\Delta t$$

Iterations are made until $\Delta t \to 0$ and the final value of t is the solution of the above equation, i.e., the current function crosses the limit $I = A_2$ at time t. Since

$$I = I_L + I_c$$

and

$$I_c = CL_k \frac{d^2 I_L}{dt^2}$$

and

$$\frac{d^2 I_L}{dt^2} = \frac{R^2 B}{4L^2} e^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) - \frac{RB\nu}{2L} e^{-\frac{R}{2L}}\cos(\nu t+\theta_2)$$

$$- \frac{RB\nu}{2L} e^{-\frac{R}{2L}t}\cos(\nu t+\theta_2) - B\nu^2 e^{-\frac{R}{2L}t}\sin(\nu t+\theta_2)$$

$$- \frac{E_m\omega^2}{LL_k\omega^2 Z}\sin(\omega t+\phi+\theta_1)$$

$$\therefore I = I_L + I_c$$

$$= A + Be^{-\frac{R}{2L}t} \sin(\nu t+\theta_2) + \frac{E_m}{LL_1\omega^2 Z} \sin(\omega t+\phi+\theta_1)$$

$$+ CL_k[\frac{R^2B}{4L^2} e^{-\frac{R}{2L}t} \sin(\nu t+\theta_2) - \frac{RB\nu}{2L} e^{-\frac{R}{2L}t} \cos(\nu t+\theta_2)$$

$$- \frac{RB\nu}{2L} e^{-\frac{R}{2L}t} \cos(\nu t+\theta_2) - B\nu^2 e^{-\frac{R}{2L}t} \sin(\nu t+\theta_2)$$

$$- \frac{E_m\omega^2}{LL_k\omega^2 Z} \sin(\omega t+\phi+\theta_1)]$$

$$\therefore I = A + Be^{-\frac{R}{2L}t} \sin(\nu t+\theta_2) + \frac{E_m}{LL_k\omega^2 Z} \sin(\omega t+\phi+\theta_1)$$

$$+ \frac{CL_kR^2B}{4L^2} e^{-\frac{R}{2L}t} \sin(\nu t+\theta_2) - \frac{CL_kRB\nu}{2L} e^{-\frac{R}{2L}t} \cos(\nu t+\theta_2)$$

$$- \frac{CL_kRB\nu}{2L} e^{-\frac{R}{2L}t} \cos(\nu t+\theta_2) - CL_kB\nu^2 e^{-\frac{R}{2L}t} \sin(\nu t+\theta_2)$$

$$- \frac{CE_m}{LZ} \sin(\omega t+\phi+\theta_1)$$

Knowing the time t_1 at which the function I_L crosses the limiting value $I_L = A_2$, it would be possible to evaluate I at time t_1 of the first piece segment of the magnetization curve as follows:

$$I = A + Be^{-\frac{R}{2L}t}\sin(\nu t+\theta_2) + \frac{E_m}{LL_1\omega^2 Z}\sin(\omega t+\phi+\theta_1)$$

$$+ \frac{CL_k R^2 B}{4L^2} e^{-\frac{R}{2L}t} \sin(\nu t+\theta_2) - \frac{CL_1 RB\nu}{2L} e^{-\frac{R}{2L}t} \cos(\nu t+\theta_2)$$

$$- \frac{CL_1 RB\nu}{2L} e^{-\frac{R}{2L}t} \cos(\nu t+\theta_2) - CL_1 B\nu^2 e^{-\frac{R}{2L}t} \sin(\nu t+\theta_2)$$

$$- \frac{CE_m}{LZ} \sin(\omega t+\phi+\theta_1)$$

Then Q_c is evaluated for the next piece segment of the magnetization curve:

$$Q_c = -\frac{CL_1 R}{2L}(B) \sin(\nu t+\theta_2)\ e^{-\frac{R}{2L}t} + (CL_1\nu)(B) e^{-\frac{R}{2L}t} \cos(\nu t+\theta_2)$$

$$+ \frac{CE_m}{L\omega Z} \cos(\omega t+\phi+\theta_1)$$

2. Evaluating A, B, and θ_2 for the second segment

Setting the limiting current $I = I_0' = A_2$ and the limiting $Q = Q_0$ for the second piece of magnetization curve $b_2 = L_2$, the procedure outlined will be repeated until the time solution is plotted for all pieces of the magnetization curve.

B. G. W. Swift's Method

For unloaded pi circuit assume that the input to nonlinearity N is λ.

$$\lambda(t) = \lambda_m \cos(\omega t+\phi) + \mu \cos \omega t \qquad \mu << \lambda_m \qquad (10.14)$$

where μ is an incremental perturbation on $\lambda(t)$, ϕ is any phase relationship between the main signal and the perturbation.

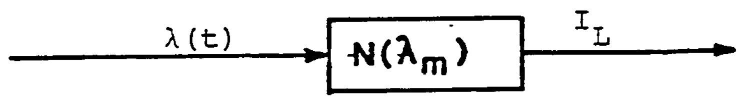
Assume that the input by passing through the nonlinearity $N(\lambda_m)$ is multiplied by a gain factor $K(\lambda_m,\phi)$ which is the fundamental component transfer function gain that depends on λ_m and ϕ. In this case, $K(\lambda_m,\phi)G(j\omega) + 1 = 0$ will be the characteristic equation, provided the solution harmonics are filtered sufficiently by the $G(j\omega)$ transfer function so that the typical linear system stability criterion can be used. Thus, the stability criterion is given by:

$$K(\lambda_m,\phi)G(j\omega) = -1$$

or

$$G(j\omega) = \frac{-1}{K(\lambda_m,\phi)}$$

where the intersection points of the LHS and RHS of this equation are the critical points. To plot $G(j\omega)$ and $K(\lambda_m,\phi)$ on the same graph in order to find these critical points, we must first solve for $K(\lambda_m,\phi)$. This is accomplished by using the diagram of Figure 8.



and $I_L = \lambda(t)+4\lambda^5(t)$. (10.15)

Using Swift's model given by $i_L = \lambda+4\lambda^5$, we find that:

$$i_L = [\lambda_m\cos(\omega t+\phi)+\mu\cos\omega t] + 4[\lambda_m\cos(\omega t+\phi)+\mu\cos\omega t]^5 \quad (10.16)$$

$$= \underbrace{\lambda_m\cos(\omega t+\phi)+\mu\cos\omega t}_{*}$$

$$+ 4[\lambda_m^2\cos^2(\omega t+\phi)+2\lambda_m\mu\cos\omega t\cos(\omega t+\phi)+\mu^2\cos^2\omega t]^2$$

$$\times \underbrace{[\lambda_m\cos(\omega t+\phi)+\mu\cos\omega t]}_{**}$$

$$= *+4[\lambda_m^4\cos^4(\omega t+\phi)+4\lambda_m^2\mu^2\cos^2\omega t\cos^2(\omega t+\phi) + \mu^4\cos^4\omega t$$

$$+ 4\lambda_m^3\mu\cos^3(\omega t+\phi)\cos\omega t + 2\mu^2\lambda_m^2\cos^2(\omega t+\phi)\cos^2\omega t$$

$$+ 4\lambda_m\mu^3\cos^3\omega t\cos(\omega t+\phi)\,[**]$$

$$= *+4[\lambda_m^5\cos^5(\omega t+\phi) + 4\lambda_m^3\mu^2\cos^3(\omega t+\phi)\cos^2\omega t$$

$$+ \mu^4\lambda_m\cos^4\omega t\cos(\omega t+\phi) + 4\lambda_m^4\mu\cos^4(\omega t+\phi)\cos\omega t$$

$$+ 2\mu^2\lambda_m^3\cos^3(\omega t+\phi)\cos^2\omega t + 4\lambda_m^2\mu^3\cos^3\omega t\cos^2(\omega t+\phi)$$

$$+ \mu\lambda_m^4\cos^4(\omega t+\phi)\cos\omega t + 4\lambda_m^2\mu^3\cos^3\omega t\cos^2(\omega t+\phi) + \mu^5\cos^5\omega t$$

$$+\ 4\lambda_m^3\mu^2\cos^2\omega t\cos^3(\omega t+\phi)\ +2\mu^3\lambda_m^2\cos^2(\omega t+\phi)\cos^3\omega t$$

$$+\ 4\lambda_m\mu^4\cos^4\omega t\cos(\omega t+\phi)\,]$$

$$\begin{aligned} i_L = & \overbrace{*+4\lambda_m^5\cos^5(\omega t+\phi)}^{1} + \overbrace{16\lambda_m^3\mu^2\cos^3(\omega t+\phi)\cos^2\omega t}^{2} \\ & + \overbrace{4\mu^4\lambda_m\cos^4\omega t\cos(\omega t+\phi)}^{3} + \overbrace{16\lambda_m^4\mu\cos^4(\omega t+\phi)\cos\omega t}^{4} \\ & + \overbrace{8\mu^2\lambda_m^3\cos^3(\omega t+\phi)\cos^2\omega t}^{5} + \overbrace{16\lambda_m^2\mu^3\cos^3\omega t\cos^2(\omega t+\phi)}^{6} \\ & + \overbrace{4\mu\lambda_m^4\cos^4(\omega t+\phi)\cos\omega t}^{7} + \overbrace{16\lambda_m^2\mu^3\cos^3\omega t\cos^2(\omega t+\phi)}^{8} \\ & + \overbrace{4\mu^5\cos^5\omega t}^{9} + \overbrace{16\lambda_m^3\mu^2\cos^2\omega t\cos^3(\omega t+\phi)}^{10} \\ & + \overbrace{8\mu^3\lambda_m^2\cos^2(\omega t+\phi)\cos^3\omega t}^{11} + \overbrace{16\lambda_m\mu^4\cos^4\omega t\cos(\omega t+\phi)}^{12} \end{aligned}$$

Combine the following terms: 2, 5 and 10; 3 and 12; 7 and 4, 6, 8 and 11.

$$\begin{aligned} \therefore\ i_L = & *+\underbrace{4\lambda_m^5\cos^5(\omega t+\phi)}_{\text{I}} + \underbrace{40\lambda_m^3\mu^2\cos^3(\omega t+\phi)\cos^2\omega t}_{\text{II}} \\ & + \underbrace{20\mu^4\lambda_m\cos^4\omega t\cos(\omega t+\phi)}_{\text{III}} + \underbrace{20\lambda_m^4\mu\cos^4(\omega t+\phi)\cos\omega t}_{\text{IV}} \\ & + \underbrace{40\lambda_m^2\mu^3\cos^3\omega t\cos^2(\omega t+\phi)}_{\text{V}} + \underbrace{4\mu^5\cos^5\omega t}_{\text{VI}} \end{aligned}$$

Using trigonometric identities and simplify.

<u>Term I</u> = $4\lambda_m^5 \cos^5(\omega t+\phi)$

$$I = 4\lambda_m^5[\frac{1}{16}\{10\cos(\omega t+\phi)+5\cos3(\omega t+\phi)+\cos5(\omega t+\phi)\}]$$

$$I = \frac{\lambda_m^5}{4}[10\cos(\omega t+\phi)+5\cos3(\omega t+\phi)+\cos5(\omega t+\phi)]$$

<u>Term II</u> = $40\lambda_m^3\mu^2\cos^3(\omega t+\phi)\cos^2\omega t$

$$II = 40\lambda_m^3\mu^2[\frac{1}{4}\{3\cos(\omega t+\phi)+\cos3(\omega t+\phi)\}\{\frac{1+\cos2\omega t}{2}\}]$$

$$= 20\lambda_m^3\mu^2[\frac{3}{4}\cos(\omega t+\phi)+\frac{3}{4}\cos(\omega t+\phi)\cos2\omega t+\frac{1}{4}\cos3(\omega t+\phi)$$
$$+\frac{1}{4}\cos3(\omega t+\phi)\cos2\omega t]$$

$$II = 15\lambda_m^3\mu^2\cos(\omega t+\phi) + 15\lambda_m^3\mu^2\cos2\omega t\cos(\omega t+\phi) + 5\lambda_m^3\mu^2\cos3(\omega t+\phi)$$
$$+ 5\lambda_m^3\mu^2\cos3(\omega t+\phi)\cos2\omega t$$

$$II = 15\lambda_m^3\mu^2\cos(\omega t+\phi) + \frac{15\lambda_m^3\mu^2}{2}[\cos3(\omega t+\frac{\phi}{3})+\cos(\omega t-\phi)]$$
$$+ 5\lambda_m^3\mu^2\cos3(\omega t+\phi) + \frac{5\lambda_m^3\mu^2}{2}[\cos5(\omega t+\frac{\phi}{5})+\cos(\omega t+\phi)]$$

$$II = \frac{35}{2}\lambda_m^3\mu^2\cos(\omega t+\phi) + \frac{15}{2}\lambda_m^3\mu^2\cos(\omega t+\frac{\phi}{3}) + \frac{5}{2}\lambda_m^3\mu^2\cos5(\omega t+\frac{\phi}{5})$$
$$+ 5\mu^2\lambda^3\cos3(\omega t+\phi) + \frac{15}{2}\lambda^3\mu^2\cos(\omega t-\phi)$$

<u>Term III</u> $=20\mu^4\lambda_m\cos^4\omega t\cos(\omega t+\phi)$

$$III = 20\mu^4\lambda_m[\tfrac{3}{8}+\tfrac{1}{2}\cos 2\omega t+\tfrac{1}{8}\cos 4\omega t]\cos(\omega t+\phi)$$

$$= \tfrac{15}{2}\mu^4\lambda_m\cos(\omega t+\phi)+10\mu^4\lambda_m\cos 2\omega t\cos(\omega t+\phi)$$

$$+ \tfrac{5}{2}\mu^4\lambda_m\cos 4\omega t\cos(\omega t+\phi)$$

$$III = \tfrac{15}{2}\mu^4\lambda_m\cos(\omega t+\phi)+5\mu^4\lambda_m[\cos 3(\omega t+\tfrac{\phi}{3})+\cos(\omega t-\phi)]$$

$$+ \tfrac{5}{4}\mu^4\lambda_m[\cos 5(\omega t+\tfrac{\phi}{5})+\cos 3(\omega t-\tfrac{\phi}{3})]$$

$$III = \tfrac{15}{2}\mu^4\lambda_m\cos(\omega t+\phi) + 5\mu^4\lambda_m\cos 3(\omega t+\tfrac{\phi}{3}) + 5\mu^4\lambda_m\cos(\omega t-\phi)$$

$$+ \tfrac{5}{4}\mu^4\lambda_m\cos 5(\omega t+\tfrac{\phi}{5}) + \tfrac{5}{4}\mu^4\lambda_m\cos 5(\omega t-\tfrac{\phi}{3})$$

<u>Term IV</u> $= 20\lambda_m^4\mu\cos^4(\omega t+\phi)\cos\omega t$

$$IV= 20\lambda_m^4\mu[\tfrac{3}{8}+\tfrac{1}{2}\cos 2(\omega t+\phi)+\tfrac{1}{8}\cos 4(\omega t+\phi)]\cos\omega t$$

$$= \tfrac{15}{2}\lambda_m^4\mu\cos\omega t + 10\lambda_m^4\mu\cos 2(\omega t+\phi)\cos\omega t + \tfrac{5}{2}\lambda_m^4\mu\cos 4(\omega t+\phi)\cos\omega t$$

$$IV = \tfrac{15}{2}\lambda_m^4\mu\cos\omega t + \frac{10\lambda_m^4\mu}{2}[\cos 3(\omega t+\tfrac{2\phi}{3})+\cos(\omega t+2\phi)]$$

$$+ \tfrac{5}{4}\lambda_m^4\mu[\cos 5(\omega t+\tfrac{4\phi}{5})+\cos 3(\omega t+\tfrac{4\phi}{3})]$$

$$IV = \frac{15}{2}\lambda_m^4\mu\cos\omega t + 5\lambda_m^4\mu\cos 3(\omega t+\frac{2\phi}{3}) + 5\lambda_m^4\mu\cos(\omega t+2\phi)$$

$$+ \frac{5}{4}\lambda^4\mu\cos 5(\omega t+\frac{4\phi}{5}) + \frac{5}{4}\lambda_m^4\mu\cos 3(\omega t+\frac{4\phi}{3})$$

Term V $= 40\lambda_m^2\mu^3\cos^3\omega t\cos^2(\omega t+\phi)$

$$V = 40\lambda_m^2\mu^3[\frac{1}{4}\{3\cos\omega t+\cos 3\omega t\}\{\frac{1+\cos 2(\omega t+\phi)}{2}\}]$$

$$V = 5\lambda_m^2\mu^3[3\cos\omega t+3\cos 2(\omega t+\phi)\cos\omega t+\cos 3\omega t+\cos 3\omega t\cos 2(\omega t+\phi)]$$

$$V = 15\lambda_m^2\mu^3\cos\omega t + \frac{15\lambda_m^2\mu^3}{2}[\cos 3(\omega t+\frac{2\phi}{3})+\cos(\omega t+\phi 2)]$$

$$+ 5\lambda_m^2\mu^3(\cos 3\omega t + \frac{5}{2}\lambda_m^2\mu^3[\cos 5(\omega t+\frac{2\phi}{5})+\cos(\omega t-2\phi)]$$

Term VI $= 4\mu^5\cos^5\omega t$

$$= 4\mu^5[\frac{1}{16}\{10\cos\omega t+5\cos 3\omega t+\cos 5\omega t\}]$$

$$= \frac{1}{4}\mu^5[10\cos\omega t+5\cos 3\omega t+\cos 5\omega t]$$

$$VI = \frac{5}{2}\mu^5\cos\omega t + \frac{5}{4}\mu^5\cos 3\omega t+\frac{1}{4}\mu^5\cos 5\omega t$$

Omitting terms in μ^2 or higher powers of μ because $\mu << \lambda_m$ and omitting terms in frequencies of 2ω or greater because G(s) is essentially low pass filter, we obtain:

$$\text{Term I} = \frac{5}{2}\lambda_m^5\cos(\omega t+\phi)$$

$$\text{Term II} = 0$$

$$\text{Term III} = 0$$

$$\text{Term IV} = \frac{15}{2}\lambda_m^4\mu\cos\omega t + 5\lambda_m^4\mu\cos(\omega t+2\phi)$$

$$\text{Term V} = 0$$

$$\text{Term VI} = 0$$

$$\therefore\ i_L = * + \text{Term I} + \text{Term II} + \text{Term III} + \text{Term IV} + \text{Term V} + \text{Term VI}$$

$$* = \lambda_m\cos(\omega t+\phi) + \mu\cos\omega t$$

$$i_L = \lambda_m\cos(\omega t+\phi) + \mu\cos\omega t + \frac{5}{2}\lambda_m^5\cos(\omega t+\phi) + \frac{15}{2}\lambda_m^4\mu\cos\omega t + 5\lambda_m^4\mu\cos(\omega t+2\phi)$$

From the definition of the sinusoidal input Describing Function $K(\lambda_m,\phi)$:

$$K(\lambda_m,\phi) = \frac{\text{Fundamental of incremental output phasor}}{\text{Incremental input phasor}}$$

$$K(\lambda_m,\phi) = \frac{\mu+\frac{15}{2}\lambda_m^4\mu + 5\lambda_m^4\mu e^{j2\phi}}{\mu}$$

$$K(\lambda_m,\phi) = 1 + \frac{15}{2}\lambda_m^4 + 5\lambda_m^4 e^{j2\phi} \qquad (10.17)$$

If

$$A = 1+\frac{15}{2}\lambda_m^4 \quad \text{and} \quad B = 5\lambda_m^4$$

$$\therefore\ K = A+Be^{j2\phi}$$

We would like to graph $(-\frac{1}{K})$

Since $K = A+Be^{j2\phi}$

$$\therefore K-A = Be^{j2\phi}$$

$$\frac{K-A}{B} = e^{j2\phi}$$

$$\therefore\ \frac{B}{K-A} = e^{-j2\phi} = M$$

$$KM-AM = B$$

$$\therefore K = \frac{B+AM}{M}$$

and

$$\frac{1}{K} = \frac{M}{B+AM} = \frac{e^{-j2\phi}}{B+Ae^{-j2\phi}} = \frac{1}{A+Be^{+j2\phi}}$$

$$\frac{1}{K} = \frac{1}{B+Ae^{-j2\phi}} \quad \text{and} \quad \left|\frac{1}{K}\right| = \left|\frac{1}{B+Ae^{-j2\phi}}\right| = \left|\frac{1}{B+Ae^{+j2\phi}}\right|$$

$$\frac{1}{K} = \left|\frac{1}{B+Ae^{+j2\phi}}\right| e^{-j2\phi}$$

Since $\frac{K-A}{B} = e^{j2\phi}$

$$\therefore = \frac{e^{-j2\phi}}{B+A\left(\frac{K-A}{B}\right)} = \frac{e^{-j2\phi}}{B+\frac{AK-A^2}{B}}$$

$$= \frac{e^{-j2\phi}}{\frac{B^2+AK-A^2}{B}}$$

$$\frac{1}{K} = \frac{Be^{-j2\phi}}{B^2+AK-A^2}$$

$$\therefore B^2+AK-A^2 = KBe^{-j2\phi}$$

$$B^2-A^2 = -AK + KBe^{-j2\phi}$$

$$B^2 - A^2 = -K(A-Be^{-j2\phi})$$

$$-\frac{1}{K} = \frac{A-Be^{-j2\phi}}{B^2-A^2} = \frac{A}{B^2-A^2} - \frac{B}{B^2-A^2}e^{-j2\phi} \qquad (10.18)$$

XI. APPENDIX C: COMPUTER PROGRAMS

The following programs are implemented in FORTRAN IV with version G, H, and WATFIV compilers:

1. Program I
   Analysis of pi-circuit for determining critical lambda and inductance.

2. Program II
   Analysis of series circuit for determining critical current and inductance.

3. Program III
   Capacitance response of pi-circuit for determining critical capacitance range.

4. Program IV
   Capacitance response of series circuit for determining critical capacitance range.

5. Program V
   Gear package to solve pi-circuit differential equation.

Program I

```
C
C
C
C
C                          ANALYSIS OF PI CIRCUIT
C
C MAIN PROGRAM CALCULATES THE INCREMENTAL DESCRIBING FUNCTION
C FOR A SINUSOIDAL INPUT,DETERMINES THE FREQUENCY RESPONSE CHARACTERIST-
C ICS OF THE LINEAER PART OF THE SYSTEM,PLOTS THEM ON A COMMON GRAPH,
C AND FINDS OUT THE CRITICAL VALUES OF LAMBDA AT WHICH FERRORESONANCE
C OCCURS.
C
C
C DEFINITION OF TERMS:                     1. DESCRIBING FUNCTION
C
C -------------------
C                     NLMDM=DESCRIBING FUNCTION OF THE INPUT AMPLITUDE.
C                     PNLMDM=DERIVATIVE OF NLMDM WITH RESPECT TO LAMBDA
C                     AMPLITUDE.
C                     LMDM=INPUT SIGNAL AMPLITUDE TO NON-LINEARITY.
C                     LMDZ=VALUE OF LAMBDA AT THE KNEE OF THE
C                     NON-LINEARITY.
C                     CCLMDM=DESCRIBING FUNCTION CIRCLE CENTER.
C                     CRLMDM=DESCRIBING FUNCTION CIRCLE RADIUS.
C                     M1=SLOPE OF 1ST. SECTION OF THE NON-LINEARITY.
C                     M2=SLOPE OF THE 2ND. SECTION OF THE NON-LINEARITY.
C                     LMD=LAMBDA
C
C
C
C
C                                          2. TRANSFER FUNCTION
C                     H(I)=TRANSFER FUNCTION.
C                     K*A1*S**0+A2*S**1+...........AN+1*S**N
C               H(S) = -------------------------------------
C                     B1*S**0+B2*S**1+............BN+1*S**N
C
C                     NA=NUMBER OF NUMERATOR COEFFICIENT N+1.
C                     NB=NUMBER OF DENOMINATOR COEFFICIENT N+1.
C                     WN=FREQUENCY NORMALIZING CONSTANT.
C                     CON=CONSTANT MULTIPYING FACTOR.
C
C
C FREQUENCY RESPONSE PLOT FOR THE LINEAR PART TRANSFER FUNCTION G(S)
C
C
C  SUB-PROGRAM CFRQRE-DETERMINES FREQUENCY
C CHARACTERISTICS GIVEN THE TRANSFER
C FUNCTION IN FACTORED OR POLYNOMIAL FORM
C REQUIRED SUBROUTINES-ZROFRQ,PLYFRQ
```

```
C
C
C
      REAL DATA1X(64),DATA1Y(64),PH60HZ,AM60HZ,RE60HZ,IM60HZ,PHASE,AMPL
      REAL LMDM(20),U(64),V(64),HLNPHS,NPHASE(64),A,B,INCR
      REAL CCLMDM(14),CRLMDM(14),SQURV,MAG(64),INDUCT(20),LLNPHS
      REAL NLMDM(20),PNLMDM(14),M1,M2,LMDZ,RATLMD,GLMD,FLMD,HLMD,ANGZ
      REAL MAGSQR,VTURAT
      INTEGER I,J,L,M,N,K1,K2,N1,N2,INDEX,INDEY,INDEZ
      CHARACTER*10 CENTER,RADIUS
      CHARACTER*15 TITLE
      CHARACTER*20 XLAB,YLAB
      CHARACTER*20 DATLAB
      DIMENSION OMEGA(100),ID(2),FW(100),NAME(15)
      DATA ID(1)/'FREQ'/
      DATA ID(2)/'OMGA'/
      COMPLEX H(100),DH(100),CONJG
C
C
C
      READ,XLAB,YLAB
1     READ(5,100,END=400)NAME
      READ(5,101)NFW,LNLGP,NPLYZ,FWEXL,FWUNC,
     1DFWPC
      NFW=NFW+1
      GO TO (2,6,10,11),LNLGP
    2 NW=(FWUNC-FWEXL)/DFWPC+1.0
      IF(NW.GT.300) NW=100
      FW(1)=FWEXL
      DO 5 I=2,NW
    5 FW(I)=FW(I-1)+DFWPC
      GO TO 11
    6 NW=FWUNC*DFWPC+1.0
      IF(NW.GT.300) NW=100
      DELEXP=1.0/DFWPC
      DO 9 I=1,NW
      FW(I)=10.0**FWEXL
    9 FWEXL=FWEXL+DELEXP
      GO TO 11
   10 READ(5,102)NW,(FW(I),I=1,NW)
   11 WRITE(6,200)
      WRITE(6,100)NAME
      DO 51 I=1,NW
      OMEGA(I)=FW(I)
      IF(NFW.EQ.1) OMEGA(I)=6.2831853*OMEGA(I)
   51 CONTINUE
```

```
      IF(NPLYZ.EQ.1)GO TO 12
      CALL ZRGFRQ(NW,OMEGA,H,DH)
      GO TO 13
   12 CALL PLYFRQ(NW,OMEGA,H,DH,DATA1X,DATA1Y)
   13 WRITE(6,201)ID(NFW)
      DO 14 I=1,NW
      R=REAL(H(I))
      DATA1X(I)=REAL(H(I))
      X=AIMAG(H(I))
      DATA1Y(I)=AIMAG(H(I))
      AMPL=X**2+R**2
      ATTEN =-10.0*ALOG10(AMPL)
      PHASE=360+57.2957795*ATAN2(X,R)
      GPDLY=(X*REAL(DH(I))-R*AIMAG(DH(I)))/AMPL
      AMPL=SQRT(AMPL)
      IF(I.EQ.35)THEN DO
          PH60HZ=PHASE
          AM60HZ=AMPL
          RE60HZ=REAL(H(I))
          IM60HZ=AIMAG(H(I))
          PRINT,'      ','PH60HZ','            ','AM60HZ','         ','RE60HZ',
     1'         ','IM60HZ'
          PRINT,'      ','______','            ','______','         ','______',
     1'         ','______'
          PRINT,PH60HZ,AM60HZ,RE60HZ,IM60HZ
      ELSE DO
          END IF
   14 WRITE(6,202)FW(I),ATTEN,PHASE,GPDLY,AMPL,R,X
C
C
      CALL GRAPH(55,DATA1X,DATA1Y,0,102,15.0,10.0,0.2,-2.0,0.2,-1.0,
     1XLAB,YLAB,'DESCRIBING FUNCTION  ;','GRAPH    1;')
C
C
      GO TO 400
  100 FORMAT(1X,13A4)
  101 FORMAT(3I1,3E15.0)
  102 FORMAT(I2/(E18.0,4E15.0))
  200 FORMAT(1H1 //42X,26HFREQUENCY ANALYSIS PROGRAM)
  201 FORMAT(//12X,A6,7X,9HATTEN(DB),6X,10HPHASE(DEG),8X,11HGROUP DELAY,12X,
     19HMAGNITUDE,18X,9HREAL PART,10X,9HIMAG PART)
  202 FORMAT(E18.8,2F16.6,4E19.6)
  400 N=14
      INDEX=0
      INDEY=0
      INDEZ=0
      READ,M1,M2,LMDZ
```

```
C
C
C
      DO 20 I=1,N
         READ,LMDM(I),TITLE
         M1=0
         M2=0
         PRINT,'LMDM(I)=',LMDM(I)
         RATLMD=LMDZ/LMDM(I)
         PRINT,'RATLMD=',RATLMD
         GLMD=1-(RATLMD)**2
         PRINT,'GLMD=',GLMD
         FLMD=SQRT(GLMD)
         PRINT,'FLMD=',FLMD
         HLMD=FLMD*RATLMD
         PRINT,'HLMD=',HLMD
         ANGZ=ARSIN(RATLMD)
         PRINT,'ANGZ=',ANGZ
C
C
C                                DESCRIBING FUNCTION
C                                -------------------
C
         NLMDM(I)=((2*(M1-M2)/3.1415927)*(ANGZ+HLMD)+M2)
         PRINT,'NLMDM(I)=',NLMDM(I)
C
C
C
C                   CALCULATION OF INDUCTANCE
C                   -------------------------
C
         INDUCT(I)=1/NLMDM(I)
         PRINT,'                                   ',TITLE,'INDUCTANCE'
         PRINT,'                                   ','-------------------',
     1'----------'
C
         PRINT,'                                   ',INDUCT(I)
C
C
C        CALCULATION OF NLMDM DERIVATIVE PNLMDM(I):
C        ------------------------------------------
C
         PNLMDM(I)=((4*(M1-M2)/3.1415927)*(1/FLMD)*(-(RATLMD)**2/(LMDZ)+
     1   (RATLMD**4)/LMDZ))
C
C
         PRINT,'PNLMDM(I)=',PNLMDM(I)
C
```

```
C
C         CALCULATION OF CIRCLE CENTER AND RADIUS OF THE DESCRIBING FN.:
C         ----------------------------------------------------------------
C
          CCLMDM(I)=+(2*NLMDM(I)+LMDM(I)*PNLMDM(I))/
     1   (((NLMDM(I))**2)*2+2*NLMDM(I)*LMDM(I)*PNLMDM(I))
C
C
          CRLMDM(I)=(LMDM(I)*PNLMDM(I))/(((NLMDM(I))**2)*2+2*NLMDM(I)*
     1   LMDM(I)*PNLMDM(I))
          PRINT,'        '
C
C
          PRINT,'                              ','CENTER',
     1'                                  ','RADIUS'
          PRINT,'                              ','------',
     1'                                  ','------'
C
C
          PRINT,'                 ',CCLMDM(I),'                    ',CRLMDM(I)
C
C
          PRINT,'        '
          PRINT,'        '
          INCR=2*CRLMDM(I)/32.
          INCR=-1*INCR
          U(1)=(CRLMDM(I)-CCLMDM(I))+INCR
C
C
          DO 333 J=1,31
              K=J+1
              U(K)=U(J)+INCR
  333         CONTINUE
C
C
C         CALCULATION OF THE INCREMENTAL DESCRIBING FUNCTION:
C         ----------------------------------------------------
C
          PRINT,'        '
          PRINT,'        '
          PRINT,'                              ','LAMBDA',I,'CIRCLE'
          PRINT,'                              ','----------------------------'
C
          M=32
          DO 15 J=1,M
               SQURV=CRLMDM(I)**2-(U(J)+CCLMDM(I))**2
               IF(SQURV.LT.0)THEN DO
                    PRINT,'LAMBDA=',LMDM(I),TITLE
                    PRINT,'----------------------'
                    PRINT,'U(J)=',U(J),'             ','SQURV=',SQURV
                    PRINT,'-----------------------','             ','-------------'
```

```
C
C              NEGATIVE SQURV IS DISCARDED FROM THIS DATA SET
C              ----------------------------------------------
               GO TO 15
            ELSE DO
               END IF
C
C
            V(J)=SQRT(SQURV)
            DATA1X(J)=U(J)
            DATA1Y(J)=V(J)
 15         CONTINUE
C
C
C        DESCRIBING FUNCTION GRAPH
C        -------------------------
         DO 17 J=1,M
            K1=J+32
            K2=33-J
            U(K1)=U(K2)
            DATA1X(K1)=U(K1)
            V(K1)=V(K2)
            V(K1)=-1*V(K1)
            DATA1Y(K1)=V(K1)
 17         CONTINUE
C
C
         PRINT,'   ','REAL PART','                        ','IMAGINERY PART',
     1'                                                      ','NPHASE',
     1'              ','MAGNITUDE'
         PRINT,'   ','_________','                        ','______________',
     1'                                                      ','______',
     1'              ','_________'
         L=64
C
C
         DO 18 J=1,L
            MAGSQR=U(J)**2+V(J)**2
            MAG(J)=SQRT(MAGSQR)
            VTURAT=V(J)/U(J)
            NPHASE(J)=180+57.2957795*ATAN(VTURAT)
            A=ABS(PHOCHZ-NPHASE(J))
            B=ABS(MAG(J)-AMCOHZ)
C
```

```
C
          PRINT,U(J),'          ',V(J),'                         ',
     1    '          ',NPHASE(J),'               ',MAG(J)
 18       CONTINUE
C
C
      LLNPHS=NPHASE(9)
      HENPHS=NPHASE(56)
C
C
      IF(PH60HZ.GT.LLNPHS.AND.PH60HZ.LT.HENPHS)THEN DO
         DO 21 K=1,L
C
            IF(MAG(K).GT.AM60HZ)THEN DO
               N1=N1+1
C
               IF(N1.EQ.64)THEN DO
                  INDEY=INDEY+1
               ELSE DO
                  END IF
C
            ELSE DO
               N2=N2+1
C
               IF(N2.EQ.64)THEN DO
                  INDEZ=INDEZ+1
               ELSE DO
                  END IF
C
               END IF
 21         CONTINUE
      ELSE DO
         INDEX=INDEX+1
         END IF
C
C
      IF(INDEX.EQ.14)THEN DO
         PRINT,'60 HZ POINT IS OUT OF ANGLE LIMIT'
         GO TO 25
      ELSE DO
         END IF
C
C
```

```
          IF(INDEY.EQ.14)THEN DO
             PRINT,'60 HZ POINT IS CLOSE TO THE ORIGIN'
             PRINT,'__________________________________________________'
          ELSE DO
             END IF
C
C
          IF(N1.EQ.64.OR.N2.EQ.64)THEN DO
             PRINT,'60 HZ POINT IS OUTSIDE DESCRIBING FUNCTION',TITLE
             PRINT,'__________________________________________________'
          ELSE DO
             END IF
C
C
          IF(N1.NE.64.AND.N1.NE.0)THEN DO
             PRINT,'60 HZ POINT IS INSIDE DESCRIBING FUNCTION',TITLE
             PRINT,'__________________________________________________'
          ELSE DO
             END IF
C
C
 25       WRITE(DATLAB,10000)I
10000     FORMAT('LAMBDA',I3,';')
C
C
          CALL GRAPH5(64,DATA1X,DATA1Y,0,106,DATLAB)
C
C
          PRINT,'        '
          PRINT,'        '
          PRINT,'        '
          PRINT,'        '
 20       CONTINUE
C
C
          IF(INDEY.EQ.14.OR.INDEZ.EQ.14.OR.(INDEY.EQ.0.AND.INDEZ.EQ.0
     1    .AND.INDEX.EQ.14))THEN DO
              PRINT,'        '
              PRINT,'        '
              PRINT,'CONCLUSION'
              PRINT,'__________'
              PRINT,'60 HZ POINT IS OUTSIDE THE NON-LINEARITY REGION.'
              PRINT,'THEREFORE FERRORESONANCE WILL NOT OCCUR.'
              PRINT,'        '
              PRINT,'        '
```

```
      ELSE DO
         PRINT,'CONCLUSION'
         PRINT,'__________'
         PRINT,'60 HZ POINT IS INSIDE THE NON-LINEARITY REGION.'
         PRINT,'THEREFORE FERRORESONANCE WILL OCCUR.'
         END IF
C
C
      READ,XLAB,YLAB
C
C
      DO 55 J=1,N
         DATA1X(J)=LMDM(J)
         DATA1Y(J)=INDUCT(J)
 55      CONTINUE
C
C
      DO 111 I=15,17
         READ,LMDM(I),INDUCT(I)
         DATA1X(I)=LMDM(I)
         DATA1Y(I)=INDUCT(I)
 111     CONTINUE
C
C
      PRINT,'       ','LAMBDA','                    ','INDUCTANCE'
      PRINT,'       ','______','                    ','__________'
C
C
      DO 57 J=1,13
         PRINT,LMDM(J),'             ',INDUCT(J)
 57      CONTINUE
C
C
      CALL GRAPH(20,DATA1X,DATA1Y,0,2,14.0,10.0,0.1,0.3,0.1,0.0,
     1XLAB,YLAB,'INDUCTANCE;','GRAPH  3;')
C
C
      STOP
      END
C
```

```
C
C SUBROUTINE ZROFRQ-READS DATA AND COMPUTES
C REAL AND IMAGINARY PARTS OF TRANSFER FUNCTION
C WHEN GIVEN IN FACTORED FORM
C
      SUBROUTINE ZROFRQ(NW,OMEGA,H,DH)
      DIMENSION OMEGA(100)
      COMPLEX H(100),DH(100),CON,W,Z(40),P(40),ZP,CONJG,CMPLX
      M=0
      N=0
      READ(5,100) NZ,NP,WN,CON
      IF(NZ.EQ.0) GO TO 2
      DO 1 K=1,NZ
      READ(5,101)I,J,ZP
      DO 1 L=1,J
      M=M+1
      Z(M)=ZP/WN
      IF(I.EQ.1) GO TO 1
      M=M+1
      Z(M)=CONJG(ZP)/WN
    1 CONTINUE
    2 DO 3 K=1,NP
      READ(5,101)I,J,ZP
      DO 3 L=1,J
      N=N+1
      P(N)=ZP/WN
      IF(I.EQ.1) GO TO 3
      N=N+1
      P(N)=CONJG(ZP)/WN
    3 CONTINUE
      CON=CON/WN**(N-M)
      WRITE(6,200)WN,CON
      IF(M.EQ.0)GO TO 4
      WRITE(6,201) (Z(I),I=1,M)
    4 WRITE(6,202) (P(I),I=1,N)
      DO 8 I=1,NW
      W=CMPLX(0.0,OMEGA(I))
      DH(I)=0.0
      H(I)=CON
      IF(M.EQ.0) GO TO 6
      DO 5 K=1,M
      H(I)=H(I)*(W-Z(K))
    5 DH(I)=DH(I)+1.0/(W-Z(K))
    6 DO 7 K=1,N
```

```
      H(I)=F(I)/(W-P(K))
    7 DH(I)=DH(I)-1.0/(W-P(K))
    8 DH(I)=(0.0,1.0)*H(I)*DH(I)
      RETURN
  100 FORMAT(2I2,E14.0,2E15.0)
  101 FORMAT(2I1,E16.0,E15.0)
  200 FORMAT(//15X,25HTHE FREQ NORM CONSTANT=,
     1E15.8,21H , THE MULT CONSTANT=,E15.8,2H+J,E15.8)
  201 FORMAT(//48X,25HTHE NUMERATOR ZEROS ARE,
     1//(E35.8,4H   +J,E16.8,E25.8,4H   +J,E16.8))
  202 FORMAT(//47X,25HTHE DENOMINATOR ZEROS ARE
     1//(E35.8,4H   +J,E16.8,E25.8,4H   +J,E16.8))
      END
C
C SUBROUTINE PLYFRQ-READS DATA AND COMPUTES
C REAL AND IMAGINARY PARTS OF TRANSFER FUNCTION
C WHEN GIVEN IN POLYNOMIAL FORM
      SUBROUTINE PLYFRQ(NW,OMEGA,H,DH,DATA1X,DATA1Y)
      CHARACTER*20 XLAB
      REAL DATA1X(55),DATA1Y(55)
      CHARACTER*20 YLAB
      DIMENSION K1(41),OMEGA(100)
      COMPLEX H(100),DH(100),DA(41),A(41),DB(41),B(41),CONJG,CMPLX,
     1CON,W,AA,DAA,BB,DBB
      DO 1 I=1,41
    1 K1(I)=I-1
      READ,XLAB,YLAB
      READ(5,100)NA,NB,WN,CON
      READ(5,101)(A(I),I=1,NA)
      READ(5,101)(B(I),I=1,NB)
      IF(NA.LT.2) GO TO 3
      DO 2 I=2,NA
      A(I)=A(I)*WN**(I-1)
    2 DA(I-1)=FLOAT(I-1)*A(I)
    3 DO 4 I=2,NB
      B(I)=B(I)*WN**(I-1)
    4 DB(I-1)=FLOAT(I-1)*B(I)
      WRITE(6,200) WN,CON
      WRITE(6,201)(A(I),K1(I),I=1,NA)
      WRITE(6,202)
```

```
      WRITE(6,201)(B(I),KI(I),I=1,NB)
      M=NA-1
      N=NB-1
      DO 8 I=1,NW
      W=CMPLX(0.0,OMEGA(I))
      AA=A(NA)
      DAA=(0.0,0.0)
      IF(NA.EQ.1) GO TO 6
      DO 5 K=1,M
      L=NA-K
      AA=AA*W+A(L)
    5 DAA=DAA*W+DA(L)
    6 AA=CCN*AA
      DAA=CCN*DAA
      BB=B(NB)
      DBB=(0.0,0.0)
      DO 7 K=1,N
      L=NB-K
      BB=BB*W+B(L)
    7 DBB=DBB*W+DB(L)
      H(I)=AA/BB
    8 DH(I)=(0.0,1.0)*(BB*DAA-AA*DBB)/(BB*BB)
      RETURN
  100 FORMAT(2I2,E14.0,2E15.0)
  101 FORMAT(E18.0,E15.0)
  200 FORMAT(//15X,23HTHE FREQ NORM CONSTANT=,
     1E15.8,21H , THE MULT CONSTANT=,
     2E15.8,2H+J,E15.8,
     3///45X,27HTHE NUMERATOR POLYNOMIAL IS)
  201 FORMAT(//(39X,E15.8,4H  +J,E15.8,5H  S**,I2))
  202 FORMAT(//44X,29HTHE DENOMINATOR POLYNOMIAL IS)
      END
$ENTRY
```

Program II

```
C
C                                  ANALYSIS USING THE SERIES CIRCUIT
C                                  ---------------------------------
C MAIN PROGRAM CALCULATES THE INCREMENTAL DESCRIBING FUNCTION
C FOR A SINUSOIDAL INPUT,DETERMINES THE FREQUENCY RESPONSE CHARACTERIST-
C ICS OF THE LINEAER PART OF THE SYSTEM,PLOTS THEM ON A COMMON GRAPH,
C AND FINDS OUT THE CRITICAL VALUES OF CURRENT AT WHICH FERRORESONANCE
C OCCURS.
C
C
C DEFINITION OF TERMS:                   1. DESCRIBING FUNCTION
C --------------------
C                    NCRTM=DESCRIBING FUNCTION OF THE INPUT AMPLITUDE.
C                    PNCRTM=DERIVATIVE OF NCRTM WITH RESPECT TO CURRENT
C                    AMPLITUDE.
C                    CRTM=INPUT SIGNAL AMPLITUDE TO NON-LINEARITY.
C                    CRTZ=VALUE OF CURRENT AT THE KNEE OF THE
C                    NON-LINEARITY.
C                    CCCRTM=DESCRIBING FUNCTION CIRCLE CENTER.
C                    CRCRTM=DESCRIBING FUNCTION CIRCLE RADIUS.
C                    M1=SLOPE OF 1ST. SECTION OF THE NON-LINEARITY.
C                    M2=SLOPE OF THE 2ND. SECTION OF THE NON-LINEARITY.
C                    CRT=CURRENT
C
C
C
C                                        2. TRANSFER FUNCTION
C                    H(S)=TRANSFER FUNCTION.
C                    K*A1*S**0+A2*S**1+...........AN+1*S**N
C            H(S) =  -------------------------------------
C                    B1*S**0+B2*S**1+............BN+1*S**N
C
C                    NA=NUMBER OF NUMERATOR COEFFICIENT N+1.
C                    NB=NUMBER OF DENOMINATOR COEFFICIENT N+1.
C                    WN=FREQUENCY NORMALIZING CONSTANT.
C                    CON=CONSTANT MULTIPYING FACTOR.
C
C
C FREQUENCY RESPONSE PLOT FOR THE LINEAR PART TRANSFER FUNCTION G(S)
C
C
C  SUB-PROGRAM CFRURE-DETERMINES FREQUENCY
C CHARACTERISTICS GIVEN THE TRANSFER
C FUNCTION IN FACTORED OR POLYNOMIAL FORM
C REQUIRED SUBROUTINES-ZROPR2,PLYRS
```

```
C
C
C
      REAL DATAIX(64),DATAIY(64),PH60HZ,AM60HZ,RE60HZ,IM60HZ,PHASE,AMPL
      REAL CRTM(20),U(64),V(64),HLNPHS,NPHASE(64),A,B,INCR
      REAL CCCRTM(20),CRCRTM(20),SQURV,MAG(64),INDUCT(20),LLNPHS
      REAL NCRTM(20),PNCRTM(20),M1,M2,CRTZ,RATCRT,GCRT,FCRT,HCRT,ANGZ
      REAL MAGSQR,VTURAT
      INTEGER I,J,L,M,N,K1,K2,N1,N2,INDEX,INDEY,INDEZ
      CHARACTER*10 CENTER,RADIUS
      CHARACTER*15 TITLE
      CHARACTER*20 XLAB,YLAB

      CHARACTER*20 DATLAB
      DIMENSION OMEGA(100),ID(2),FW(100),NAME(13)
      DATA ID(1)/'FREQ'/
      DATA ID(2)/'OMGA'/
      COMPLEX H(100),DH(100),CONJG
C
C
C
      READ,XLAB,YLAB
1     READ(5,100,END=400)NAME
      READ(5,101)NFW,LNLGP,NPLYZ,FWEXL,FWUNC,
     1DFWPC
      NFW=NFW+1
      GO TO (2,6,10,11),LNLGP
    2 NW=(FWUNC-FWEXL)/DFWPC+1.0
      IF(NW.GT.300) NW=100
      FW(1)=FWEXL
      DO 5 I=2,NW
    5 FW(I)=FW(I-1)+DFWPC
      GO TO 11
    6 NW=FWUNC*DFWPC+1.0
      IF(NW.GT.300) NW=100
      DELEXP=1.0/DFWPC
      DO 9 I=1,NW
      FW(I)=10.0**FWEXL
    9 FWEXL=FWEXL+DELEXP
      GO TO 11
   10 READ(5,102)NW,(FW(I),I=1,NW)
   11 WRITE(6,200)
      WRITE(6,100)NAME
      DO 21 I=1,NW
```

```
      OMEGA(I)=FW(I)
      IF(NFW.EQ.1) OMEGA(I)=6.2831853*OMEGA(I)
   51 CONTINUE
      IF(NPLYZ.EQ.1)GO TO 12
      CALL ZROFRQ(NW,OMEGA,H,DH)
      GO TO 13
   12 CALL PLYFRQ(NW,OMEGA,H,DH,DATA1X,DATA1Y)
   13 WRITE(6,201)ID(NFW)
      DO 14 I=1,NW
      R=REAL(H(I))
      DATA1X(I)=REAL(H(I))
      X=AIMAG(H(I))
      DATA1Y(I)=AIMAG(H(I))
      AMPL=X**2+R**2
      ATTEN =-10.0*ALOG10(AMPL)
      PHASE=57.2957795*ATAN2(X,R)
      GPDLY=(X*REAL(DH(I))-R*AIMAG(DH(I)))/AMPL
      AMPL=SQRT(AMPL)
      IF(I.EQ.35)THEN DO
          PH60HZ=PHASE
          AM60HZ=AMPL
          RE60HZ=REAL(H(I))
          IM60HZ=AIMAG(H(I))
          PRINT,'        ','PH60HZ','              ','AM60HZ','              ','RE60HZ',
     1'          ','IM60HZ'
          PRINT,'        ','______','              ','______','              ','______',
     1'          ','______'
          PRINT,PH60HZ,AM60HZ,RE60HZ,IM60HZ
      ELSE DO

          END IF
   14 WRITE(6,202)FW(I),ATTEN,PHASE,GPDLY,AMPL,R,X
C
C
      CALL GRAPH(55,DATA1X,DATA1Y,0,102,20.0,10.0,10.0,-100.,10.,-50.,
     1XLAB,YLAB,'DESCRIBING FUNCTION   ;','GRAPH    1;')
C
C
      GO TO 400
  100 FORMAT(1X,13A4)
  101 FORMAT(3I1,3E15.0)
  102 FORMAT(I2/(E18.0,4E15.0))
  200 FORMAT(1H1 //42X,26HFREQUENCY ANALYSIS PROGRAM)
  201 FORMAT(//12X,A6,7X,9HATTEN(DB),6X,10HPHASE(DEG),5X,11HGROUP DELAY,32X,
     19HMAGNITUDE,13X,9HREAL PART,10X,9HIMAG PART)
```

```
202 FORMAT(E18.6,2F10.6,4E19.6)
400 N=15
    INDEX=0
    INDEY=0
    INDEZ=0
    READ,M1,M2,CRTZ
C
C
C
    DO 20 I=1,N
       READ,CRTM(I),TITLE
       N1=0
       N2=0
       PRINT,'CRTM(I)=',CRTM(I)
       RATCRT=CRTZ/CRTM(I)
       PRINT,'RATCRT=',RATCRT
       GCRT=1-(RATCRT)**2
       PRINT,'GCRT=',GCRT
       FCRT=SQRT(GCRT)
       PRINT,'FCRT=',FCRT
       HCRT=FCRT*RATCRT
       PRINT,'HCRT=',HCRT
       ANGZ=ARSIN(RATCRT)
       PRINT,'ANGZ=',ANGZ
C
C
C                                  DESCRIBING FUNCTION
C                                  -------------------
C
       NCRTM(I)=((2*(M1-M2)/3.1415927)*(ANGZ+HCRT)+M2)
       PRINT,'NCRTM(I)=',NCRTM(I)
C
C
C
C                        CALCULATION OF INDUCTANCE
C                        -------------------------
C
       INDUCT(I)=NCRTM(I)
       PRINT,'                                        ',TITLE,'INDUCTANCE'
       PRINT,'                                        ','-------------------',
     1'----------'
C
       PRINT,'                                         ',INDUCT(I)
C
C
```

```
C         CALCULATION OF NCRTM DERIVATIVE PNCRTM(I):
C         -----------------------------------------
C
          PNCRTM(I)=((4*(M1-M2)/3.1415927)*(1/PCRT)*(-(RATCRT)**2/(CRT2)+
     1    (RATCRT**4)/CRT2))
C
C
          PRINT,'PNCRTM(I)=',PNCRTM(I)
C
C
C         CALCULATION OF CIRCLE CENTER AND RADIUS OF THE DESCRIBING FN.:
C         ------------------------------------------------------------
C
          CCCRTM(I)=+(2*NCRTM(I)+CRTM(I)*PNCRTM(I))/
     1   (((NCRTM(I)))**2)*2+2*NCRTM(I)*CRTM(I)*PNCRTM(I))
C
C
          CRCRTM(I)=(CRTM(I)*PNCRTM(I))/(((NCRTM(I)))**2)*2+2*NCRTM(I)*
     1   CRTM(I)*PNCRTM(I))*(-1)
          PRINT,'      '
C
C
          PRINT,'                         ','CENTER',
     1'                 ','RADIUS'
          PRINT,'                         ','------',
     1'                 ','------'
C
C
          PRINT,'              ',CCCRTM(I),'                       ',CRCRTM(I)
C
C
          INCR=2*CRCRTM(I)/32.
          INCR=-1*INCR
          U(1)=(CRCRTM(I)-CCCRTM(I))+INCR
C
C
          DO 333 J=1,31
             K=J+1
             U(K)=U(J)+INCR
  333        CONTINUE
```

```
C
C
C          CALCULATION OF THE INCREMENTAL DESCRIBING FUNCTION:
C          ----------------------------------------------------
C
           PRINT,'      '
           PRINT,'      '
           PRINT,'                        ','CURRENT',I,'CIRCLE'
           PRINT,'                        ','___________________________'
C
           M=32
           DO 15 J=1,M
              SQURV=CRCRTM(I)**2-(U(J)+CCCRTM(I))**2
              IF(SQURV.LT.0)THEN DO
                 PRINT,'CURRENT=',CRTM(I),TITLE
                 PRINT,'____________________'
                 PRINT,'U(J)=',U(J),'             ','SQURV=',SQURV
                 PRINT,'____________________','        ','____________'
C
C                NEGATIVE SQURV IS DISCARDED FROM THIS DATA SET


C                ------------------------------------------------
                 GO TO 15
              ELSE DO
                 END IF
C
C
              V(J)=SQRT(SQURV)
              DATA1X(J)=U(J)
              DATA1Y(J)=V(J)
 15           CONTINUE
C
C
C          DESCRIBING FUNCTION GRAPH
C          -------------------------
           DO 17 J=1,M
              K1=J+32
              K2=33-J
              U(K1)=U(K2)
              DATA1X(K1)=U(K1)
              V(K1)=V(K2)
              V(K1)=-1*V(K1)
              DATA1Y(K1)=V(K1)
 17           CONTINUE
```

```
C
C
      PRINT,'     ','REAL PART','                        ','IMAGINERY PART',
     1'                                                    ','NPHASE',
     1'                ','MAGNITUDE'
      PRINT,'     ','__________','                       ','________________',
     1'                                                    ','______',
     1'                ','__________'
      L=64
C
C
      DO 18 J=1,L
         MAGSQR=U(J)**2+V(J)**2
         MAG(J)=SQRT(MAGSQR)
         VTURAT=V(J)/U(J)
         NPHASE(J)=180+57.2957795*ATAN(VTURAT)
         A=ABS(PH60HZ-NPHASE(J))
         B=ABS(MAG(J)-AM60HZ)
C
C
         IF(A.LE.0.1.AND.B.LE.0.001)THEN DO
            PRINT,'         '
            PRINT,'         '
            PRINT,'         '
            PRINT,'CRITICAL CURRENT=',CRTM(I)
            PRINT,'___________________________________'
            PRINT,'         '
            PRINT,'         '
            PRINT,'         '
         PRINT,U(J),'          ',V(J),'                              ',
     1   '          ','NPHASE(J),'              ',MAG(J)
            INDEY=INDEY-1
            GO TO 25
         ELSE DO
            END IF
C
C


         PRINT,U(J),'          ',V(J),'                              ',
     1   '          ','NPHASE(J),'              ',MAG(J)
 18      CONTINUE
```

```
C
C
      LLNPHS=NPHASE(9)
      HLNPHS=NPHASE(56)
C
C
      IF(PH60HZ.GT.LLNPHS.AND.PH60HZ.LT.HLNPHS)THEN DO
         DO 21 K=1,L
C
            IF(MAG(K).GT.AM60HZ)THEN DO
               N1=N1+1
C
               IF(N1.EQ.64)THEN DO
                  INDEY=INDEY+1
               ELSE DO
                  END IF
C
            ELSE DO
               N2=N2+1
C
               IF(N2.EQ.64)THEN DO
                  INDEZ=INDEZ+1
               ELSE DO
                  END IF
C
               END IF
 21         CONTINUE
      ELSE DO
         INDEX=INDEX+1
         END IF
C
C
      IF(INDEX.EQ.15)THEN DO
         PRINT,'60 HZ POINT IS OUT OF ANGLE LIMIT'
         GO TO 25
      ELSE DO
         END IF
C
C
      IF(INDEY.EQ.15)THEN DO
         PRINT,'60 HZ POINT IS CLOSE TO THE ORIGIN'
         PRINT,'--------------------------------------------------------'
      ELSE DO
         END IF
```

```
C
C
          IF(N1.EQ.64.OR.N2.EQ.64)THEN DO
             PRINT,'60 HZ POINT IS OUTSIDE DESCRIBING FUNCTION',TITLE
             PRINT,'______________________________________________________'
          ELSE DO
             END IF
C
C
          IF(N1.NE.64.AND.N1.NE.0)THEN DO
             PRINT,'60 HZ POINT IS INSIDE DESCRIBING FUNCTION',TITLE
             PRINT,'______________________________________________________'
          ELSE DO
             END IF
C
C
 25       WRITE(DATLAB,10000)I
10000     FORMAT('CURRENT',I3,';')
C
C
          CALL GRAPHS(64,DATA1X,DATA1Y,0,100,DATLAB)
C
C
          PRINT,'      '
          PRINT,'      '
          PRINT,'      '
          PRINT,'      '
 20       CONTINUE
C
C
          IF(INDEY.EQ.15.OR.INDEZ.EQ.15.OR.(INDEY.EQ.0.AND.INDEZ.EQ.0
     1    .AND.INDEX.EQ.15))THEN DO
             PRINT,'      '
             PRINT,'      '
             PRINT,'CONCLUSION'
             PRINT,'__________'
             PRINT,'60 HZ POINT IS OUTSIDE THE NON-LINEARITY REGION.'
             PRINT,'THEREFORE FERRORESONANCE WILL NOT OCCUR.'
             PRINT,'      '
             PRINT,'      '
          ELSE DO
             PRINT,'CONCLUSION'
             PRINT,'__________'
             PRINT,'60 HZ POINT IS INSIDE THE NON-LINEARITY'REGION.'
             PRINT,'THEREFORE FERRORESONANCE WILL OCCUR.'
             END IF
```

```
C
C
      READ,XLAB,YLAB
C
C
      DO 55 J=1,N
         DATA1X(J)=CRTM(J)
         DATA1Y(J)=INDUCT(J)
 55      CONTINUE
C
C
      DO 111 I=13,15
         READ,CRTM(I),INDUCT(I)
         DATA1X(I)=CRTM(I)
         DATA1Y(I)=INDUCT(I)
 111     CONTINUE
C
C
      PRINT,'        ','CURRENT','                    ','INDUCTANCE'
      PRINT,'        ','_______','                    ','__________'
C
C
      DO 57 J=1,13
         PRINT,CRTM(J),'             ',INDUCT(J)
 57      CONTINUE
C
C


      CALL GRAPH(20,DATA1X,DATA1Y,0,2,10.0,10.0,4.0,0.0,0.1,0.0,
     1XLAB,YLAB,'INDUCTANCE;','GRAPH   3;')
C
C
      STOP
      END
C
C
C SUBROUTINE ZROFRQ-READS DATA AND COMPUTES
C REAL AND IMAGINARY PARTS OF TRANSFER FUNCTION
C WHEN GIVEN IN FACTORED FORM
C
C
```

```
      SUBROUTINE ZROFRQ(NW,OMEGA,H,DH)
      DIMENSION OMEGA(100)
      COMPLEX H(100),DH(100),CON,W,Z(40),P(40),ZP,CONJG,CMPLX
      M=0
      N=0
      READ(5,100) NZ,NP,WN,CON
      IF(NZ.EQ.0) GO TO 2
      DO 1 K=1,NZ
      READ(5,101)I,J,ZP
      DO 1 L=1,J
      M=M+1
      Z(M)=ZP/WN
      IF(I.EQ.1) GO TO 1
      M=M+1
      Z(M)=CONJG(ZP)/WN
    1 CONTINUE
    2 DO 3 K=1,NP
      READ(5,101)I,J,ZP
      DO 3 L=1,J
      N=N+1
      P(N)=ZP/WN
      IF(I.EQ.1) GO TO 3
      N=N+1
      P(N)=CONJG(ZP)/WN
    3 CONTINUE
      CON=CON/WN**(N-M)
      WRITE(6,200)WN,CON
      IF(M.EQ.0)GO TO 4
      WRITE(6,201) (Z(I),I=1,M)
    4 WRITE(6,202) (P(I),I=1,N)
      DO 8 I=1,NW
      W=CMPLX(0.0,OMEGA(I))
      DH(I)=0.0
      H(I)=CON
      IF(M.EQ.0) GO TO 6
      DO 5 K=1,M
      H(I)=H(I)*(W-Z(K))
    5 DH(I)=DH(I)+1.0/(W-Z(K))
    6 DO 7 K=1,N
      H(I)=H(I)/(W-P(K))
    7 DH(I)=DH(I)-1.0/(W-P(K))
    8 DH(I)=(0.0,1.0)*H(I)*DH(I)
      RETURN
  100 FORMAT(2I2,E14.0,2E15.0)
  101 FORMAT(2I1,E16.0,E15.0)
  200 FORMAT(//15X,23HTHE FREQ NORM CONSTANT=,
     1E15.8,21H   THE MULT CONSTANT=,E15.8,3H J,E15.8)
```

```
  201 FORMAT(//48X,23HTHE NUMERATOR ZEROS ARE,
     1//(E35.8,4H   +J,E16.8,E25.8,4H   +J,E16.8))
  202 FORMAT(//47X,25HTHE DENOMINATOR ZEROS ARE
     1//(E35.8,4H   +J,E16.8,E25.8,4H   +J,E16.8))
      END
C
C SUBROUTINE PLYFRQ-READS DATA AND COMPUTES
C REAL AND IMAGINARY PARTS OF TRANSFER FUNCTION
C WHEN GIVEN IN POLYNOMIAL FORM
      SUBROUTINE PLYFRQ(NW,OMEGA,H,DH,DATA1X,DATA1Y)
      CHARACTER*20 XLAB
      REAL DATA1X(55),DATA1Y(55)
      CHARACTER*20 YLAB
      DIMENSION K1(41),OMEGA(100)
      COMPLEX H(100),DH(100),DA(41),A(41),DB(41),B(41),CONJG,CMPLX,
     1CON,W,AA,DAA,BB,DBB
      DO 1 I=1,41
    1 K1(I)=I-1
      READ,XLAB,YLAB
      READ(5,100)NA,NB,WN,CON
      READ(5,101)(A(I),I=1,NA)
      READ(5,101)(B(I),I=1,NB)
      IF(NA.LT.2) GO TO 3
      DO 2 I=2,NA
      A(I)=A(I)*WN**(I-1)
    2 DA(I-1)=FLOAT(I-1)*A(I)
    3 DO 4 I=2,NB
      B(I)=B(I)*WN**(I-1)
    4 DB(I-1)=FLOAT(I-1)*B(I)
      WRITE(6,200) WN,CON
      WRITE(6,201)(A(I),K1(I),I=1,NA)
      WRITE(6,202)
      WRITE(6,201)(B(I),K1(I),I=1,NB)
      M=NA-1
      N=NB-1
      DO 8 I=1,NW
      W=CMPLX(0.0,OMEGA(I))
      AA=A(NA)
      DAA=(0.0,0.0)
```

```
      IF(NA.EQ.1) GO TO 6
      DO 5 K=1,M
      L=NA-K
      AA=AA*W+A(L)
    5 DAA=DAA*W+DA(L)
    6 AA=CON*AA
      DAA=CON*DAA
      BB=E(NB)
      DBB=(0.0,0.0)
      DO 7 K=1,N
      L=NB-K
      BB=BB*W+B(L)
    7 DBB=DBB*W+DB(L)
      H(I)=AA/BB
    8 DF(I)=(0.0,1.0)*(BB*DAA-AA*DBB)/(BB*BB)
      RETURN
  100 FORMAT(2I2,E14.0,2E15.0)
  101 FORMAT(E18.0,E15.0)
  200 FORMAT(//15X,23HTHE FREQ NORM CONSTANT=,
     1E15.8,21H , THE MULT CONSTANT=,
     2E15.8,2H+J,E15.8,
     3//45X,27HTHE NUMERATOR POLYNOMIAL IS)
  201 FORMAT(//(39X,E15.8,4H  +J,E15.8,5H  S**,I2))
  202 FORMAT(//44X,29HTHE DENOMINATOR POLYNOMIAL IS)
      END
$ENTRY
```

Program III

```
C                                    ANALYSIS OF PI CIRCUIT
C MAIN PROGRAM CALCULATES THE INCREMENTAL DESCRIBING FUNCTION
C FOR A SINUSOIDAL INPUT.
C PROGRAM DTERMINES THE MAGNITUDE OF THE TRANSFER FUNCTION AS A FUNCTION
C OF THE LINEAR CAPACITANCE. THE REAL AND THE IMAGINERY PARTS OF THE
C TRANSFER FUNCTION ARE THEN PLOTTED AND THE PLOT IS USED TO DETERMINE
C THE CRITICAL VALUES OF CAPACITANCE TO PLACE THE SYSTEM IN FR.
C
C
C DEFINITION OF TERMS:                        1. DESCRIBING FUNCTION
C -------------------
C
C
C                         NLMDM=DESCRIBING FUNCTION OF THE INPUT AMPLITUDE.
C                         PNLMDM=DERIVATIVE OF NLMDM WITH RESPECT TO LAMBDA
C                         AMPLITUDE.
C                         LMDM=INPUT SIGNAL AMPLITUDE TO NON-LINEARITY.
C                         LMDZ=VALUE OF LAMBDA AT THE KNEE OF THE
C                         NON-LINEARITY.
C                         CCLMDM=DESCRIBING FUNCTION CIRCLE CENTER.
C                         CRLMDM=DESCRIBING FUNCTION CIRCLE RADIUS.
C                         M1=SLOPE OF 1ST. SECTION OF THE NON-LINEARITY.
C                         M2=SLOPE OF THE 2ND. SECTION OF THE NON-LINEARITY.
C
C
C
C                                             2. TRANSFER FUNCTION
C
C
C                         LML=LAMBDA
C DEFINITION OF TERMS:
C -------------------
C
C                        L= LINEAR INDUCTANCE
C                        R= RESISTANCE
C                        C= CAPACITANCE IN PER UNIT
C                        C1=CAPACITANCE
C                        RE(G)= +S1
C                        IMAG(G)=+S2
C
C
C EQUATION USED:
C ------------
C
```

```
C    REG(RLC)=((L-(R**2)*C(J)-(L**2)*C(J))/((R**2)*(C(J)**2)+(1-L*C)**2))
C
C     IMG(R,L,C)=(-R/((R**2)*(C**2)+(1-L*C)**2))
C
C
      REAL DATA1X(64),DATA1Y(64),PHASE,AMPL,NPHASE(64)
      REAL LMDM(20),U(64),V(64),INCR
      REAL CCLMDM(14),CRLMDM(14),SQURV,MAG(64)
      REAL NLMDM(20),FNLMDM(14),M1,M2,LMDZ,RATLMD,GLMD,FLMD,HLMD,ANGZ
      REAL MAGSQR,VTURAT
      INTEGER I,J,M,N,K1,K2,N1,N2
      CHARACTER*10 CENTER,RADIUS
      CHARACTER*20 XLAB,YLAB
      CHARACTER*20 DATLAB
      REAL R,L,S1,S2,C(50),C1(50)
      CHARACTER*27 TITLE
C
C
C
C
      READ,TITLE
      READ,XLAB,YLAB
      READ,R
      READ,L
      PRINT,'                          ',TITLE
      PRINT,'                          ','_________________________'
C
C
      PRINT,'                           ','RESISTANCE=',R
      PRINT,'                           ','______________________________'
C
C
      PRINT,'                           ',' INDUCTANCE=',L
      PRINT,'                           ','______________________________'
      PRINT,'      '
      PRINT,'      '
      DO 444 J=1,50
         READ,C(J)
         C1(J)=C(J)*0.7460791
         S1=((L-(R**2)*C(J)-(L**2)*C(J))/((R**2)*(C(J)**2)+
     1         (1-L*C(J))**2))
         DATA1X(J)=S1
         S2=(-R/((R**2)*(C(J)**2)+(1-L*C(J))**2))
         DATA1Y(J)=S2
 444     CONTINUE
```

```
      PRINT,'          '
      PRINT,'          '
         PRINT,'     ','  C(I)  IN MF','                    ','  REAL  ',
     1'                 ','IMAG60HZ'
         PRINT,'     ','___________','                    ','_________',
     1'                 ','________'
      PRINT,'          '
      DO 555 I=1,50
         PRINT,C1(I),'                ',DATAIX(I),'                ',DATAIY(I)
 555     CONTINUE
      PRINT,'          '
C
      CALL GRAPH(50,DATAIX,DATAIY,0,102,15.0,10.0,0.2,-2.0,0.2,
     1   -1.0,XLAB,YLAB,'CRITICAL VALUES;',' INDUCTANCE 1;')
      N=14
      READ,M1,M2,LMDZ
      DO 20 I=1,N
         READ,LMDM(I),TITLE
C
         PRINT,'LMDM(I)=',LMDM(I)
         PRINT,'          '
         RATLMD=LMDZ/LMDM(I)
C
         PRINT,'RATLMD=',RATLMD
C
         GLMD=1-(RATLMD)**2
C
         PRINT,'GLMD=',GLMD
C
         FLMD=SQRT(GLMD)
         PRINT,'FLMD=',FLMD
C
         HLMD=FLMD*RATLMD
         PRINT,'HLMD=',HLMD
C
         ANG2=ARSIN(RATLMD)
C
C                                  CALCULATION OF NLMDM
C                                  --------------------
C
C
         NLMDM(I)=((2*(M1-M2)/3.1415927)*(ANG2+HLMD)+M2)
C
         PRINT,'NLMDM(I)=',NLMDM(I)
```

```
C
C
C         CALCULATION OF NLMDM DERIVATIVE PNLMDM(I):
C         ---------------------------------------
C
          PNLMDM(I)=((4*(M1-M2)/3.1415927)*(1/FLMD)*(-(RATLMD)**2/(LMDZ)+
     1    (RATLMD**4)/LMDZ))
C
C
          PRINT,'PNLMDM(I)=',PNLMDM(I)
C
C
C         CALCULATION OF CIRCLE CENTER AND RADIUS OF THE DESCRIBING FN.:
C         ---------------------------------------------------------
          PRINT,'ANGZ=',ANGZ
C
          CCLMDM(I)=+(2*NLMDM(I)+LMDM(I)*PNLMDM(I))/
     1   (((NLMDM(I))**2)*2+2*NLMDM(I)*LMDM(I)*PNLMDM(I))
C
C
          CRLMDM(I)=(LMDM(I)*PNLMDM(I))/(((NLMDM(I))**2)*2+2*NLMDM(I)*
     1   LMDM(I)*PNLMDM(I))
          PRINT,'          '
C
C
          PRINT,'                         ','CENTER',
     1'                              ','RADIUS'
          PRINT,'                         ','______',
     1'                              ','______'
C
C
          PRINT,'          ',CCLMDM(I),'                    ',CRLMDM(I)
C
C
          PRINT,'          '
          PRINT,'          '
          INCR=((CCLMDM(I)+CRLMDM(I))-(CCLMDM(I)-CRLMDM(I)))/32
          INCR=-1*INCR
          U(1)=(CRLMDM(I)-CCLMDM(I))+INCR
          DO 333 J=1,31
             K=J+1
             U(K)=U(J)+INCR
  333        CONTINUE
```

```
C         CALCULATION OF THE INCREMENTAL DESCRIBING FUNCTION:
C         ----------------------------------------------------
C
          PRINT,'        '
          PRINT,'        '
          PRINT,'                    ','LAMBDA',I,'CIRCLE'
          PRINT,'                    ','-----------------------------'
C
          M=32
          DO 15 J=1,M
             SQURV=CRLMDM(I)**2-(U(J)+CCLMDM(I))**2
             IF(SQURV.LT.0)THEN DO
                PRINT,'       '
C
                PRINT,'       '
                PRINT,'LAMBDA=',LMDM(I),TITLE
                PRINT,'--------------------'
C
                PRINT,'U(J)=',U(J),'          ','SQURV=',SQURV
                PRINT,'--------------------','          ','-----------'
C
C
C               NEGATIVE SQURV IS DISCARDED FROM THIS DATA SET
C               ----------------------------------------------
                PRINT,'       '
                PRINT,'       '
                GO TO 15
             ELSE DO
                END IF
C
C
             V(J)=SQRT(SQURV)
             DATA1X(J)=U(J)
             DATA1Y(J)=V(J)
  15         CONTINUE
C
C
C
C         DESCRIBING FUNCTION GRAPH
C         -------------------------
```

```
      DO 17 J=1,M
         K1=J+32
         K2=33-J
         U(K1)=U(K2)
         DATA1X(K1)=U(K1)
         V(K1)=V(K2)
         V(K1)=-1*V(K1)
         DATA1Y(K1)=V(K1)
17       CONTINUE
      PRINT,'    ','REAL PART','                        ','IMAGINERY PART',
     1'                                                  ','NPHASE',
     1'                  ','MAGNITUDE'
      PRINT,'    ','_________','                        ','______________',
     1'                                                  ','______',
     1'                  ','_________'
      N1=64
      DO 18 J=1,N1
         MAGSQR=U(J)**2+V(J)**2
         MAG(J)=SQRT(MAGSQR)
         VTORAT=V(J)/U(J)
         NPHASE(J)=180+57.2957795*ATAN(VTORAT)
         PRINT,U(J),'          ',V(J),'                    ',
     1'          ',NPHASE(J),'          ',MAG(J)
18       CONTINUE
25    WRITE(DATLAB,10000)I
10000 FORMAT('LAMBDA',I3,';')
      CALL GRAPHS(64,DATA1X,DATA1Y,0,100,DATLAB)
20    CONTINUE
     STOP
     END
$ENTRY
```

Program IV

```
C                          ANALYSIS OF SERIES CIRCUIT
C MAIN PROGRAM CALCULATES THE INCREMENTAL DESCRIBING FUNCTION
C FOR A SINUSOIDAL INPUT.
C PROGRAM DTERMINES THE MAGNITUDE OF THE TRANSFER FUNCTION AS A FUNCTION
C OF THE LINEAR CAPACITANCE. THE REAL AND THE IMAGINERY PARTS OF THE
C TRANSFER FUNCTION ARE THEN PLOTTED AND THE PLOT IS USED TO DETERMINE
C THE CRITICAL VALUES OF CAPACITANCE TO PLACE THE SYSTEM IN FR.
C
C
C DEFINITION OF TERMS:                       1. DESCRIBING FUNCTION
C --------------------
C
C
C                    NCRTM=DESCRIBING FUNCTION OF THE INPUT AMPLITUDE.
C                    FNCRTM=DERIVATIVE OF NCRTM WITH RESPECT TO CURRENT
C                    AMPLITUDE.
C                    CRTM=INPUT SIGNAL AMPLITUDE TO NON-LINEARITY.
C                    CRTZ=VALUE OF CURRENT AT THE KNEE OF THE
C                    NON-LINEARITY.
C                    CCCRTM=DESCRIBING FUNCTION CIRCLE CENTER.
C                    CRCRTM=DESCRIBING FUNCTION CIRCLE RADIUS.
C                    M1=SLOPE OF 1ST. SECTION OF THE NON-LINEARITY.
C                    M2=SLOPE OF THE 2ND. SECTION OF THE NON-LINEARITY.
C
C
C
C
C                                            2. TRANSFER FUNCTION
C
C                   L= LINEAR INDUCTANCE
C                   R= RESISTANCE
C                   C= CAPACITANCE IN PER UNIT
C                   C1=CAPACITANCE
C                   RL(G)= +S1
C                   IMAG(G)=+S2
C
C
C
C EQUATION USED:
C --------------
C
```

```
C REG(R,L,C)=(L*(C**2)-C)/(R**2*C**2+(1-L*C)**2))
C IMAG(R,L,C)=(C**2*R)/(R**2*C**2+(1-L*C)**2))
C
C
C
C
      REAL DATA1X(64),DATA1Y(64),PHASE,AMPL,NPHASE(64)
      REAL CRTM(20),U(64),V(64),INCR
      REAL CCCRTM(20),CRCRTM(20),SQURV,MAG(64)
      REAL NCRTM(20),PNCRTM(20),M1,M2,CRTZ,RATCRT,GCRT,FCRT,FCRT,ANG2
      REAL MAGSQR,VTURAT
      INTEGER I,J,M,N,K1,K2,N1,N2
      CHARACTER*10 CENTER,RADIUS
      CHARACTER*20 XLAB,YLAB
      CHARACTER*20 DATLAB
      REAL R,L,S1,S2,C(51),C1(51)
      CHARACTER*27 TITLE
C
C
C
C
      READ,TITLE
      READ,XLAB,YLAB
      READ,R
      READ,L
      PRINT,'                           ',TITLE
      PRINT,'                           ','---------------------------'
C
C
      PRINT,'                             ','RESISTANCE=',R
      PRINT,'                             ','-------------------------------'
C
C
      PRINT,'                             ',' INDUCTANCE=',L
      PRINT,'                             ','-------------------------------'
      PRINT,'     '
      PRINT,'     '
      DO 444 J=1,51
         READ,C(J)
         C1(J)=C(J)*0.7460791
         S1=(L*(C(J)**2)-C(J))/((R**2)*(C(J)**2)+(1-L*C(J))**2)
         S2=(R*(C(J)**2))/((R**2)*(C(J)**2)+(1-L*C(J))**2)
         DATA1X(J)=S1
         DATA1Y(J)=S2
 444     CONTINUE
```

```
      PRINT,'        '
      PRINT,'        '
          PRINT,'      ','  C(I) IN MF','                           ','  REAL   ',
     1'                     ','IMAG60HZ'
          PRINT,'      ','____________','                           ','_________',
     1'                     ','_________'
      PRINT,'        '
      DO 555 I=1,51
          PRINT,C1(I),'                 ',DATA1X(I),'                        ',DATA1Y(I)
 555      CONTINUE
      PRINT,'        '
C
      CALL GRAPH(51,DATA1X,DATA1Y,0,102,20.0,10.0,10.0,-100.,10.,-50.,
     1XLAB,YLAB,'CRITICAL VALUES;','INDUCTANCE 1;')
      N=19
      READ,M1,M2,CRTZ
      DO 20 I=1,N
          READ,CRTM(I),TITLE
C
          PRINT,'CRTM(I)=',CRTM(I)
          PRINT,'      '
          RATCRT=CRTZ/CRTM(I)
C
          PRINT,'RATCRT=',RATCRT
C
          GCRT=1-(RATCRT)**2
C
          PRINT,'GCRT=',GCRT
C
          FCRT=SQRT(GCRT)
          PRINT,'FCRT=',FCRT
C
          HCRT=FCRT*RATCRT
          PRINT,'HCRT=',HCRT
C
          ANGZ=ARSIN(RATCRT)
C
C                                     CALCULATION OF NCRTM
C                                     --------------------
C
C
          NCRTM(I)=((2.*(M1-M2)/3.1415927)*(ANGZ+FCRT)+M2)
C
```

```
      PRINT,'NCRTM(I)=',NCRTM(I)
C
C
C     CALCULATION OF NCRTM DERIVATIVE PNCRTM(I):
C     ----------------------------------------
C
      PNCRTM(I)=((4*(M1-M2)/3.1415927)*(1/FCRT)*(-(RATCRT)**2/(CRTZ)+
     1 (RATCRT**4)/CRTZ))
C
C
      PRINT,'PNCRTM(I)=',PNCRTM(I)
C
C
C     CALCULATION OF CIRCLE CENTER AND RADIUS OF THE DESCRIBING FN.:
C     --------------------------------------------------------------
      PRINT,'ANGZ=',ANGZ
C
      CCCRTM(I)=+(2*NCRTM(I)+CRTM(I)*PNCRTM(I))/
     1 (((NCRTM(I))**2)*2+2*NCRTM(I)*CRTM(I)*PNCRTM(I))
C
C
      CRCRTM(I)=(CRTM(I)*PNCRTM(I))/(((NCRTM(I))**2)*2+2*NCRTM(I)*
     1 CRTM(I)*PNCRTM(I))*(-1)
      PRINT,'          '
C
C
      PRINT,'                                ','CENTER',
     1'                                          ','RADIUS'
      PRINT,'                                ','------',
     1'                                          ','------'
C
C
      PRINT,'               ',CCCRTM(I),'                    ',CRCRTM(I)
C
C
      PRINT,'          '
      PRINT,'          '
      INCR=((CCCRTM(I)+CRCRTM(I))-(CCCRTM(I)-CRCRTM(I)))/32
      INCR=-1*INCR
      U(1)=(CRCRTM(I)-CCCRTM(I))+INCR
      DO 333 J=1,31
        K=J+1
        U(K)=U(J)+INCR
333     CONTINUE
```

```
C
C         CALCULATION OF  THE INCREMENTAL DESCRIBING FUNCTION:
C         ---------------------------------------------------
C
          PRINT,'       '
          PRINT,'       '
          PRINT,'                       ','CURRENT',I,'CIRCLE'

          PRINT,'                       ','__________________________'
C
          M=32
          DO 15 J=1,M
              SQURV=CRCRTM(I)**2-(U(J)+CCCRTM(I))**2
              IF(SQURV.LT.0)THEN DO
                  PRINT,'       '
C
                  PRINT,'       '
                  PRINT,'CURRENT=',CRTM(I),TITLE
                  PRINT,'___________________'
C
                  PRINT,'U(J)=',U(J),'            ','SQURV=',SQURV
                  PRINT,'___________________','         ','___________'
C
C
C                 NEGATIVE SQURV IS DISCARDED FROM THIS DATA SET
C                 ----------------------------------------------
                  PRINT,'       '
                  PRINT,'       '
                  GO TO 15
              ELSE DO
                  END IF
C
C
              V(J)=SQRT(SQURV)
              DATA1X(J)=U(J)
              DATA1Y(J)=V(J)
   15         CONTINUE
C
C
C
```

```
C          DESCRIBING FUNCTION GRAPH
C          ------------------------
           DO 17 J=1,M
              K1=J+32
              K2=33-J
              U(K1)=U(K2)
              DATA1X(K1)=U(K1)
              V(K1)=V(K2)
              V(K1)=-1*V(K1)
              DATA1Y(K1)=V(K1)
 17           CONTINUE
           PRINT,'   ','REAL PART','                         ','IMAGINERY PART',
     1'                                                         ','NPHASE',
     1'                    ','MAGNITUDE'
           PRINT,'   ','_________','                         ','______________',
     1'                                                         ','______',
     1'                    ','_________'
           N1=64
           DO 18 J=1,N1
              MAGSQR=U(J)**2+V(J)**2
              MAG(J)=SQRT(MAGSQR)
              VTURAT=V(J)/U(J)
              NPHASE(J)=180+57.2957795*ATAN(VTURAT)
              PRINT,U(J),'            ',V(J),'                        ',
     1  '                ',NPHASE(J),'             ',MAG(J)
 18           CONTINUE
C
 25        WRITE(DATLAB,10000)I
10000      FORMAT('CURRENT',I3,'; ')
           CALL GRAPHS(64,DATA1X,DATA1Y,0,100,DATLAB)
 20        CONTINUE
        STOP
        END
$ENTRY
```

Program V

```
C
C PROGRAM FOR GEAR PACKAGE TO SOLVE PI CIRCUIT DIFFERENTIAL EQUATIONS.
C
      COMMON/BLK1/DATA1X,DATA1Y,DATA2Y
      COMMON /GEAR9/ HUSED,NQUSED,NSTEP,NFE,NJE
      COMMON /GEAR2/ YMAX(3)
      COMMON /GEAR3/ ERROR(3)
      COMMON /GEAR4/ SAVE1(3)
      COMMON /GEAR5/ SAVE2(3)
      COMMON /GEAR6/ PW(9)
      COMMON /GEAR7/ IPIV(3)
      REAL*8 Y(3,6)
      REAL*8 Y0(3)
      REAL*8 T0,H0,HUSED,TOUT,EPS
      REAL*8 R,L,C,E
      COMMON/CONST/R,L,C,E
      REAL*8 DATA1X(1600),DATA1Y(1600),DATA2Y(1600)
      REAL XLAB(3),YLAB(3)
      INTEGER I
      INTEGER DATLAB
      INTEGER TITLE(5)
      I=1
      READ (5,80) TITLE
80    FORMAT(10A4)
      READ (5,90) XLAB,YLAB
90    FORMAT(10A4)
      N=3
      T0=0.0
      H0=0.004411D0
      EPS=0.000001D0
      MF=21
      Y0(1)=0.0D0
      Y0(2)=1.0D0
      Y0(3)=0.0D0
      R=0.002537D0
      L=0.0335069D0
      C=46.911916D0
      E=0.2608696D0
      DT=0.18D0
      TOUT=DT+T0
      INDEX=1
      TMAX=251.2D0
```

```
10    CALL DRIVE(N,T0,H0,Y0,TOUT,EPS,MF,INDEX)
      WRITE(6,20)TOUT,Y0(1),Y0(2),Y0(3),NQUSED,HUSED,INDEX
20    FORMAT(4H T =,E12.4,7H Y0(1)=,E12.4,7H Y0(2)=,E12.4,7H Y0(3)=,
     1        E12.4,7H  NQ = ,I3,4H H =,E12.4,8H INDEX =,I3)
      IF(INDEX.EQ.0) GO TO 40
      WRITE(6,30)INDEX
30    FORMAT(//26H ERROR RETURN WITH INDEX =,I3//)
      GO TO 50
40    T0=TOUT
      DATA1X(I)=TOUT
      DATA1Y(I)=Y0(1)
      DATA2Y(I)=Y0(2)
      TOUT=TOUT+DT
      I=I+1
      IF(TOUT.LE.TMAX) GO TO 10
      NPTS=I-1
50    WRITE(6,60)NSTEP,NFE,NJE
60    FORMAT(//21H PROBLEM COMPLETED IN,I5,6H STEPS/,
     1        21X,I5,14H F EVALUATIONS/,
     1        21X,I5,14H J EVALUATIONS///)
      CALL GRAPH(-NPTS,DATA1X,DATA1Y,0,2,80.0,10.0,3.14,0.0,1.0,-5.0,
     1XLAB,YLAB,'LAMBDA VS. ANGLE;','GEAR;')
      READ (5,11) TITLE
11    FORMAT(10A4)
      READ (5,12) XLAB,YLAB
12    FORMAT(10A4)
      CALL GRAPH(-NPTS,DATA1X,DATA2Y,0,2,80.0,10.0,3.14,0.0,1.0,-5.0,
     1XLAB,YLAB,'VOLTAGE VS. ANGLE;','GEAR;')
      CALL EXIT
      END
      SUBROUTINE DIFFUN (N,T,Y,YDOT)
      REAL*8 Y(N),YDOT(N)
      REAL*8 R,L,C,E,T
      COMMON/CONST/R,L,C,E
      COMMON/BLK1/DATA1X,DATA1Y,DATA2Y
      REAL*8 DATA1X(1600),DATA1Y(1600),DATA2Y(1600)
      YDOT(1)=Y(2)
      YDOT(2)=Y(3)
      YDOT(3)=(-R*Y(1)-(L+1)*Y(2)-R*C*Y(3)+E*DSIN(T) -
     1        20*L*Y(2)*Y(1)**4-4*R*Y(1)**5)/(L*C)
      RETURN
```

```
       END
       SUBROUTINE PEDERV(N,T,Y,PD,NO)
       REAL*8 Y(NO)
       REAL*8 PD(NO,NO)
       REAL*8 R,L,C,E
       COMMON/CONST/R,L,C,E
       PD(1,2)=1.0D0
       PD(1,3)=0.0D0
       PD(2,1)=0.0D0
       PD(2,2)=0.0D0
       PD(2,3)=1.0D0
       PD(3,1)=-R/(L*C)-80*(1/C)*(Y(1)**3)*Y(2)-20*(Y(1)**4)*R/(L*C)
       PD(3,2)=-(L+1)/(L*C)-20*(1/C)*Y(1)**4
       PD(3,3)=-1*(R/L)
       RETURN
       END
//LKED.SYSLIB DD
//             DD DSN=PROG.GEAR2,DISP=SHR,UNIT=DISK,VOL=SER=LIB002
//             DD DSN=SYS1.MATHLIB,DISP=SHR
//GO.SYSIN DD *
 LAMBDA VS.  RADIANS;
RADIANS                        LAMBDA
VOLTAGE VS. RADIANS;
RADIANS                        VOLTAGE
RADIANS                        VOLTAGE
//GO.FT14F001 DD DSNAME=&&M,UNIT=SCRTCH,DISP=(NEW,PASS),
// SPACE=(800,(120,15)),DCB=(RECFM=VBS,LRECL=796,BLKSIZE=800)
//SMPLTTR EXEC PLOT,PLOTTER=INCRMNTL
//
```

XII. APPENDIX D: DERIVATION OF THE TWO-SLOPE INCREMENTAL INPUT DESCRIBING FUNCTION

A. Pi Circuit

The Describing Function is a complex gain which at fundamental frequency (60 Hz) modifies the amplitude and phase of the input to the nonlinearity.

$$N(\lambda_m,\omega) = \frac{\text{Phasor representation of output component at frequency } \omega}{\text{Phasor representation of input component at frequency } \omega}$$

Now, if the Describing Function has an in-phase and quadrature components and a continuous first derivative with respect to λ_m, then:

$$N(\lambda_m) = N_p(\lambda_m) + jN_q(\lambda_m) \tag{12.1}$$

$$\therefore \text{Output} = \text{Input}\ [N_p(\lambda_m + \mu\cos\phi) + jN_q(\lambda_m + \mu\cos\phi)] \tag{12.2}$$

$$= (\text{Input})(N_p) + jN_q\ \text{Input}$$

$$= (\text{Input})(N_p) + j\omega.\frac{N_q}{\omega}\ \text{Input}$$

$$= (\text{Input})(N_p) + \frac{N_q}{\omega}\frac{d}{dt}\ (\text{Input})$$

Substituting Equation 4.26

$$\text{Output} = (\lambda_m+\mu\cos\phi)[N_p(\lambda_m+\mu\cos\phi)]\sin(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ \frac{N_q(\lambda_m+\mu\cos\phi)}{\omega}(\lambda_m+\mu\cos\phi)\omega\cos(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$\text{Output} = (\lambda_m+\mu\cos\phi)[N_p(\lambda_m)+\mu\cos\phi\,\frac{dN_p(\lambda_m)}{d\lambda_m}]\sin(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ [N_q(\lambda_m)+\mu\cos\phi\,\frac{dN_q(\lambda_m)}{d\lambda_m}][\lambda_m+\mu\cos\phi]\cos(\omega t+\frac{\mu}{\lambda_m}\sin\phi)$$

$$\text{Output} = (\lambda_m+\mu\cos\phi)N_p(\lambda_m)\sin(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)\mu\cos\phi\,\frac{dN_p}{d\lambda_m}\sin(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)N_q(\lambda_m)\cos(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)\mu\cos\phi\,\frac{dN_q}{d\lambda_m}\cos(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$\text{Output} = (\lambda_m+\mu\cos\phi)N_p(\lambda_m)\sin(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi\,\sin(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)N_q(\lambda_m)\cos(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)\mu\cos\phi\,\dot{N}_q\cos(\omega t + \frac{\mu}{\lambda_m}\sin\phi)$$

$$\text{Output} = (\lambda_m+\mu\cos\phi)N_p(\lambda_m)[\sin\omega t\ \cos(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ \cos\ \omega t\ \sin(\frac{\mu}{\lambda_m}\ \sin\phi)]$$

$$+ (\lambda_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi[\sin\ \omega t\ \cos(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ \cos\ \omega t\ \sin(\frac{\mu}{\lambda_m}\sin\phi)]$$

$$+ (\lambda_m+\mu\cos\phi)N_q[\cos\ \omega t\ \cos(\frac{\mu}{\lambda_m}\sin\phi)$$

$$- \sin\ \omega t\ \sin(\frac{\mu}{\lambda_m}\sin\phi)]$$

$$+ (\lambda_m+\mu\cos\phi)\mu\cos\phi\dot{N}_q[\cos\ \omega t\ \cos(\frac{\mu}{\lambda_m}\sin\phi)$$

$$- \sin\ \omega t\ \sin(\frac{\mu}{\lambda_m}\sin\phi)]$$

$$\text{Output} = (\lambda_m+\mu\cos\phi)N_p\sin\ \omega t\ \cos(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)N_p\cos\ \omega t\ \sin(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi\ \sin\ \omega t\ \cos(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi\ \cos\ \omega t\ \sin(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)N_q\cos\ \omega t\ \cos(\frac{\mu}{\lambda_m}\sin\phi)$$

$$- (\lambda_m+\mu\cos\phi)N_q\sin\ \omega t\ \sin(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\ \cos\phi)\mu\cos\phi\ \dot{N}_q\ \cos\ \omega t\ \cos(\frac{\mu\sin\phi}{\lambda_m})$$

$$- (\lambda_m+\mu\cos\phi)\mu\cos\phi\ \dot{N}_q\sin\ \omega t\ \sin(\frac{\mu}{\lambda_m}\sin\phi)$$

Since $\cos(\frac{\mu}{\lambda_m}\sin\phi) = 1$

and $\sin(\frac{\mu}{\lambda_m}\sin\phi) = \frac{\mu}{\lambda_m}\sin\phi$

$$\text{Output} = (\lambda_m+\mu\cos\phi)N_p\sin\omega t + (\lambda_m+\mu\cos\phi)N_p\cos\omega t(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi\ \sin\omega t$$

$$+ (\lambda_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi\ \cos\omega t(\frac{\mu}{\lambda_m}\sin\phi)$$

$$+ (\lambda_m+\mu\cos\phi)N_q\cos\omega t - (\lambda_m+\mu\cos\phi)N_q\sin\omega t(\frac{\mu\sin\phi}{\lambda_m})$$

$$+ (\lambda_m+\mu\cos\phi)\mu\cos\phi\ \dot{N}_q\cos\omega t$$

$$- (\lambda_m+\mu\cos\phi)\mu\cos\phi\ \dot{N}_q\ \sin\omega t(\frac{\mu}{\lambda_m}\sin\phi)$$

$$\text{Output} = \lambda_m N_p\sin\omega t + \mu N_p\cos\phi\ \sin\omega t + N_p\mu\sin\phi\ \cos\omega t$$

$$+ \frac{N_p}{\lambda_m}\mu^2\sin\phi\ \cos\phi\ \cos\omega t + \lambda_m\dot{N}_p\mu\cos\phi\ \sin\omega t$$

$$+ \mu^2\cos^2\phi\sin\omega t\ \dot{N}_p$$

$$+ \lambda_m\dot{N}_p\frac{\mu^2}{\lambda_m}\cos\phi\ \cos\omega t\ \sin\phi + \frac{\mu^3}{\lambda_m}\dot{N}_p\cos^2\phi\ \sin\phi\ \cos\omega t$$

$$+ \lambda_m N_q\cos\omega t + \mu N_q\cos\phi\ \cos\omega t - \mu N_q\sin\phi\ \sin\omega t$$

$$- \frac{N_q\mu^2}{\lambda_m}\cos\phi\ \sin\phi\ \sin\omega t + \lambda_m\mu\cos\phi\ \dot{N}_q\cos\omega t$$

$$+ \mu^2\cos^2\phi\dot{N}_q\cos\omega t$$

$$- \lambda_m\ \dot{N}_q\ \frac{\mu^2}{\lambda_m}\ \cos\phi\ \sin\omega t\ \sin\phi - \frac{\dot{N}_q\mu^3}{\lambda_m}\cos^2\phi\ \sin\phi\ \sin\omega t$$

Neglecting all terms which include μ^2 or μ^3:

$$\text{Output} = \lambda_m N_p \sin \omega t + \mu N_p \cos\phi \sin \omega t + N_p \mu \sin\phi \cos \omega t$$
$$+ \lambda_m \dot{N}_p \mu \cos\phi \sin \omega t + \lambda_m N_q \cos \omega t + \mu N_q \cos\phi \cos \omega t$$
$$- \mu N_q \sin\phi \sin \omega t + \lambda_m \mu \cos\phi \, \dot{N}_q \cos \omega t$$

$$\text{Output} = \lambda_m N_p \sin \omega t + \lambda_m N_q \cos \omega t + \mu N_p \sin(\omega t+\phi)$$
$$+ \mu N_q \cos(\omega t+\phi) + \lambda_m \mu \dot{N}_p \cos\phi \sin \omega t$$
$$+ \lambda_m \mu \cos\phi \, \dot{N}_q \cos \omega t$$

$$\text{Output} = \lambda_m N_p \sin \omega t + \lambda_m N_q \cos \omega t$$
$$+ \mu[N_p \sin(\omega t+\phi) + N_q \cos(\omega t+\phi) + \lambda_m \dot{N}_p \cos\phi \sin \omega t$$
$$+ \lambda_m \cos\phi \, \dot{N}_q \cos \omega t]$$

$$\text{Incremental Describing Function} = \frac{\text{Phasor representation of output component at frequency } \omega}{\text{Phasor representation of input component at frequency } \omega}$$

$$= N(\lambda_m, \phi)$$

$$\Delta\text{Output} = \mu[N_p \sin(\omega t+\phi) + N_q \cos(\omega t+\phi) + \lambda_m \dot{N}_p \cos\phi \sin \omega t$$
$$+ \lambda_m \cos\phi \dot{N}_q \cos \omega t]$$

$$\frac{\Delta\text{Output}}{\Delta\text{Input}} = \frac{\mu[N_p \sin(\omega t+\phi) + N_q \cos(\omega t+\phi) + \lambda_m \dot{N}_p \cos\phi \sin \omega t + \lambda_m \cos\phi \, \dot{N}_q \cos \omega t]}{\mu \sin(\omega t+\phi)}$$

$$N(\lambda_m,\phi) = \frac{N_p e^{j(\phi+\omega t)} + N_q e^{j(\omega t+\phi+\frac{\pi}{2})} + \lambda_m \dot{N}_p \cos\phi\, e^{j\omega t} + \lambda_m \dot{N}_q e^{j(\omega t+\frac{\pi}{2})}\cos\phi}{e^{j(\omega t+\phi)}}$$

$$= \frac{N_p e^{j\phi} + N_q e^{j(\phi+\frac{\pi}{2})} + \lambda_m \dot{N}_p \cos\phi + \lambda_m \dot{N}_q e^{j\frac{\pi}{2}}\cos\phi}{e^{j\phi}}$$

$$= \frac{N_p e^{j\phi} + N_q e^{j\phi}\cdot e^{j\frac{\pi}{2}} + \lambda_m \dot{N}_p \cos\phi + \lambda_m \cos\phi\ \dot{N}_q \cdot e^{j\frac{\pi}{2}}}{e^{j\phi}}$$

$$= \frac{N_p e^{j\phi} + jN_q e^{j\phi} + \lambda_m \dot{N}_p \cos\phi + j\lambda_m \cos\phi\ \dot{N}_q}{e^{j\phi}}$$

$$= (N_p + jN_q) + (\lambda_m \dot{N}_p \cos\phi + j\lambda_m \cos\phi\ \dot{N}_q)e^{-j\phi}$$

$$= (N(\lambda_m) + \lambda_m \cos\phi[\dot{N}_p + j\dot{N}_q]e^{-j\phi}$$

$$= N(\lambda_m) + \lambda_m \cos\phi[\dot{N}\ (\lambda_m)]e^{-j\phi}$$

$$= N(\lambda_m) + \lambda_m \left(\frac{e^{j} + e^{-j\phi}}{2}\right)\dot{N}\ (\lambda_m)e^{-j\phi}$$

$$N(\lambda_m,\phi) = N(\lambda_m) + \frac{\lambda_m \dot{N}(\lambda_m)}{2}(1+e^{-2j\phi}) \qquad (12.3)$$

This is the Incremental Input Describing Function based on two-slope magnetization curve Describing Function of the pi circuit.

B. Series Circuit

If the describing function has an in-phase and quadrature component and a continuous first derivative with respect to I_m, then:

$$N(I_m) = N_p(I_m) + jN_q(I_m) \tag{12.4}$$

$$\text{Output} = \text{Input}\ [N_p(I_m + \mu\cos\phi) + jN_q(I_m + \mu\cos\phi)] \tag{12.5}$$

$$\text{Output} = (\text{Input})(N_p) + \frac{N_q}{\omega}\frac{d}{dt}(\text{Input})$$

Substituting equation 4.28 in the above:

$$\text{Output} = (I_m+\mu\cos\phi)[N_p(I_m+\mu\cos\phi)]\sin(\omega t + \frac{\mu}{I_m}\sin\phi)$$
$$+ \frac{N_q(I_m+\mu\cos\phi)}{\omega}(I_m+\mu\cos\phi)\,\omega\cos(\omega t + \frac{\mu}{I_m}\sin\phi)$$

$$\text{Output} = (I_m+\mu\cos\phi)[N_p(I_m)+\mu\cos\phi\frac{dN_p}{dI_m}]\sin(\omega t + \frac{\mu}{I_m}\sin\phi)$$
$$+ [N_q(I_m)+\mu\cos\phi\ \frac{dN_q}{dI_m}][I_m+\mu\cos\phi]\cos(\omega t + \frac{\mu}{I_m}\sin\phi)$$

$$\text{Output} = (I_m+\mu\cos\phi)N_p\sin(\omega t + \frac{\mu}{I_m}\sin\phi)$$
$$+ (I_m+\mu\cos\phi)\mu\cos\phi\ \frac{dN_p}{dI_m}\sin(\omega t + \frac{\mu}{I_m}\sin\phi)$$
$$+ (I_m+\mu\cos\phi)N_q\cos(\omega t + \frac{\mu}{I_m}\sin\phi)$$
$$+ (I_m+\mu\cos\phi)\mu\cos\phi\ \frac{dN_q}{dI_m}\cos(\omega t + \frac{\mu}{I_m}\sin\phi)$$

$$\text{Output} = (I_m + \mu\cos\phi)N_p \sin(\omega t + \frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m + \mu\cos\phi)\dot{N}_p \mu\cos\phi \sin(\omega t + \frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m + \mu\cos\phi)N_q \cos(\omega t + \frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m + \mu\cos\phi)\mu\cos\phi \dot{N}_q \cos(\omega t + \frac{\mu}{I_m}\sin\phi)$$

$$\text{Output} = (I_m + \mu\cos\phi)N_p [\sin \omega t \cos(\frac{\mu}{I_m}\sin\phi) + \cos \omega t \sin(\frac{\mu}{I_m}\sin\phi)]$$

$$+ (I_m + \mu\cos\phi)\dot{N}_p \mu\cos\phi [\sin \omega t \cos(\frac{\mu}{I_m}\sin\phi)$$

$$+ \cos \omega t \sin(\frac{\mu}{I_m}\sin\phi)]$$

$$+ (I_m + \mu\cos\phi)N_q [\cos \omega t \cos(\frac{\mu}{I_m}\sin\phi)$$

$$- \sin \omega t \sin(\frac{\mu}{I_m}\sin\phi)]$$

$$+ (I_m + \mu\cos\phi)\mu\cos\phi \dot{N}_q [\cos \omega t \cos(\frac{\mu}{I_m}\sin\phi)$$

$$- \sin \omega t \sin(\frac{\mu}{I_m}\sin\phi)]$$

$$\text{Output} = (I_m + \mu\cos\phi)N_p \sin \omega t \cos(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m + \mu\cos\phi)N_p \cos \omega t \sin(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m + \mu\cos\phi)\dot{N}_p \mu\cos\phi \sin \omega t \cos(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m + \mu\cos\phi)\dot{N}_p \mu\cos\phi \cos \omega t \sin(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m+\mu\cos\phi)N_q\cos\omega t\cos(\frac{\mu}{I_m}\sin\phi)$$

$$- (I_m+\mu\cos\phi)N_q\sin\omega t\sin(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m+\mu\cos\phi)\mu\cos\phi\ \dot{N}_q\cos\omega t\cos(\frac{\mu}{I_m}\sin\phi)$$

$$- (I_m+\mu\cos\phi)\mu\cos\phi\ \dot{N}_q\sin\omega t\sin(\frac{\mu}{I_m}\sin\phi)$$

Since $\cos(\frac{\mu}{I_m}\sin\phi) = 1$

and $\sin(\frac{\mu}{I_m}\sin\phi) = \frac{\mu}{I_m}\sin\phi$

$$\text{Output} = (I_m+\mu\cos\phi)N_p\sin\omega t + (I_m+\mu\cos\phi)N_p\cos\omega t(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi\sin\omega t$$

$$+ (I_m+\mu\cos\phi)\dot{N}_p\mu\cos\phi\cos\omega t(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m+\mu\cos\phi)N_q\cos\omega t - (I_m+\mu\cos\phi)N_q\sin\omega t(\frac{\mu}{I_m}\sin\phi)$$

$$+ (I_m+\mu\cos\phi)\mu\cos\phi\ \dot{N}_q\cos\omega t$$

$$- (I_m+\mu\cos\phi)\mu\cos\phi\ \dot{N}_q\sin\omega t(\frac{\mu}{I_m}\sin\phi)$$

$$\text{Output} = I_mN_p\sin\omega t + \mu N_p\cos\phi\sin\omega t + N_p\mu\sin\phi\cos\omega t$$

$$+ \frac{N_p\mu^2}{I_m}\sin\phi\cos\phi\cos\omega t + I_m\dot{N}_p\cos\phi\sin\omega t$$

$$+ \mu^2\cos^2\phi\sin\omega t\ \dot{N}_p$$

$$+ I_m\dot{N}_p\frac{\mu^2}{I_m}\cos\phi\cos\omega t\sin\phi + \frac{3}{I_m}\dot{N}_p\cos^2\phi\sin\phi\cos\omega t$$

$$N(I_m,\phi) = IDF$$

$$N(I_m,\phi) = \frac{\Delta Output}{\Delta Input}$$

$$N(I_m,\phi) = \frac{\mu[N_p \sin(\omega t+\phi) + N_q \cos(\omega t+\phi) + I_m \dot{N}_q \cos\phi \sin \omega t + I_m \cos\phi\ \dot{N}_q \cos \omega t]}{\mu \sin(\omega t+\phi)}$$

$$N(I_m,\phi) = \frac{N_p e^{j(\phi+\omega t)} + N_q e^{j(\omega t+\phi+\frac{\pi}{2})} + I_m \dot{N}_p \cos\phi\ e^{j\omega t} + I_m \cos\phi\ \dot{N}_q e^{j(\omega t+\frac{\pi}{2})}}{e^{j(\omega t+\phi)}}$$

$$= \frac{N_p e^{j\phi} + N_q e^{j(\phi+\frac{\pi}{2})} + I_m \dot{N}_p \cos\phi + I_m \cos\phi\ \dot{N}_q e^{j\frac{\pi}{2}}}{e^{j\phi}}$$

$$= \frac{N_p e^{j\phi} + N_q e^{j\phi} \cdot e^{j\frac{\pi}{2}} + I_m \dot{N}_p \cos\phi + I_m \cos\phi\ \dot{N}_q e^{j\frac{\pi}{2}}}{e^{j\phi}}$$

$$= \frac{N_p e^{j\phi} + jN_q e^{j\phi} + I_m \dot{N}_p \cos\phi + jI_m \cos\phi\ \dot{N}_q}{e^{j\phi}}$$

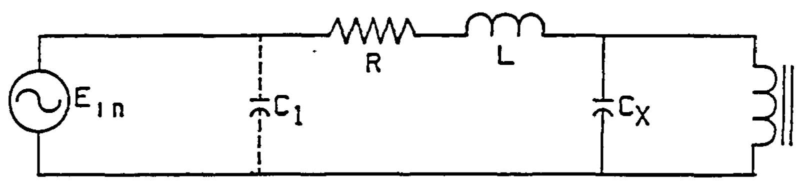
$$= N_p + jN_q + (I_m \dot{N}_p \cos\phi + jI_m \cos\phi\ \dot{N}_q) e^{-j\phi}$$

$$= N(I_m) + I_m \cos\phi\ \dot{N}(I_m) e^{-j\phi}$$

$$= N(I_m) + I_m \left(\frac{e^{j\phi}+e^{-j\phi}}{2}\right) \dot{N}(I_m) e^{-j\phi}$$

$$= N(I_m) + \frac{I_m \dot{N}(I_m)}{2}(1+e^{-2j\phi}) \qquad (12.6)$$

XIII. APPENDIX E: SOLUTION OF THE PI-CIRCUIT EQUATIONS USING JUMP POINTS



$$\frac{d\lambda(t)}{dt} = \frac{d\lambda(t)}{di_L(t)} \cdot \frac{di_L(t)}{dt} \tag{13.1}$$

$$L_{DF} = \frac{1}{N(\lambda_m)} = \frac{d\lambda(t)}{di_L(t)} \tag{13.2}$$

and

$$\frac{d\lambda(t)}{dt} = L_{DF}\frac{di_L(t)}{dt} \tag{13.3}$$

<u>Laplace transformed equations</u>:

$$S\lambda(S) = L_{DF}SI_L(S)$$

$$\therefore \lambda(S) = L_{DF}I_L(S)$$

Using Equation 3.8

$$\lambda(S) = \frac{E_{in}(S)}{S(1+RCS+LCS^2)} - \frac{R+LS}{S(1+RCS+LCS^2)}\, I_L(S)$$

or

$$S\lambda(S) = \frac{E_{in}(S)}{1+RCS+LCS^2} - \frac{R+LS}{1+RCS+LCS^2} I_L(S)$$

$$\therefore V_L(S) = S\lambda(S) = \frac{E_{in}(S)}{1+RLS+LCS^2} - \frac{R+LS}{1+RCS+LCS^2}\left(\frac{1}{L_{DF}}\lambda(S)\right)$$

$$V_L(S) = \frac{E_{in}(S)}{1+RCS+LCS^2} - \frac{(R+LS)}{L_{DF}(1+RCS+LCS^2)}\lambda(S) \qquad (13.4)$$

$$= \frac{E_{in}(S)}{LC(\frac{1}{LC} + \frac{RC}{LC}S+S^2)} - \frac{(R+LS)\lambda(S)}{L_{DF}LC(\frac{1}{LC} + \frac{RC}{LC}S+S^2)}$$

$$= \frac{\frac{E_{in}(S)}{LC}}{S^2 + \frac{R}{L}S + \frac{1}{LC}} - \frac{(\frac{R+LS}{L_{DF}LC})\lambda(S)}{S^2 + \frac{R}{L}S + \frac{1}{LC}}$$

$$V_L(S) = \frac{\frac{E_{in}(S)}{LC}}{(S+\frac{R}{2L})^2 - \frac{R^2}{4L^2} + \frac{1}{LC}} - \frac{\frac{R+LS}{L_{DF}LC}\lambda(S)}{(S+\frac{R}{2L})^2 - \frac{R^2}{4L^2} + \frac{1}{LC}}$$

$$V_L(S) = \frac{\frac{E_{in}(S)}{LC}}{(S+\frac{R}{2L})^2 + K_1^2} - \frac{\frac{R+LS}{L_{DF}LC}\lambda(S)}{(S+\frac{R}{2L})^2 + K_1^2}$$

where

$$K_1^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$$

Since $e_{in}(t) = E_m \sin \omega t$

$$\therefore \quad e_{in}(t) = E_{in}(S) = E_m\left(\frac{\omega}{S^2+\omega^2}\right)$$

and

$$\lambda(t) = \lambda_{DF} \sin \omega t$$

$$L\lambda(t) = \lambda(S) = \lambda_{DF}\left(\frac{\omega}{S^2+\omega^2}\right)$$

$$\therefore V_L(S) = \frac{\frac{E_m\omega}{LC}}{(S^2+\omega^2)[(S+\frac{R}{2L})^2+K_1^2]} - \frac{\frac{R+LS}{L_{DF}LC}\lambda_{DF}\omega}{(S^2+\omega^2)[(S+\frac{R}{2L})^2+K_1^2]}$$

$$= \frac{\frac{E_m\omega}{LC} - \frac{(R+LS)\lambda_{DF}\omega}{L_{DF}LC}}{(S^2+\omega^2)[(S+\frac{R}{2L})^2+K_1^2]}$$

$$V_L(S) = \frac{\left(\frac{E_m\omega}{LC} - \frac{R\lambda_{DF}\omega}{L_{DF}LC}\right) - \frac{L\lambda_{DF}\omega}{L_{DF}LC}S}{(S^2+\omega^2)[(S+\frac{R}{2L})^2+K_1^2]}$$

$$V_L(S) = \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} - \frac{L\lambda_{DF}\omega}{L_{DF}LC}S}{(S^2+\omega^2)[(S+\frac{R}{2L})^2+K_1^2]} = \frac{\alpha_1+\alpha_2 S}{S^2+\omega^2} + \frac{\alpha_3+\alpha_4 S}{(S+\frac{R}{2L})^2+K_1^2}$$

$$V_L(S)(S^2+\omega^2)\Big|_{s=-j\omega} = \frac{\alpha_1+\alpha_2\ S}{S^2+\omega^2}(S^2+\omega^2) + \frac{(\alpha_3+\alpha_4 S)(S^2+\omega^2)}{(S+\frac{R}{2L})^2 + K_1^2}$$

$$= \alpha_1 - j\omega\alpha_2 + \frac{(\alpha_3 - j\omega\alpha_4)(j^2\omega^2+\omega^2)}{(-j\omega+\frac{R}{2L})^2 + K_1^2}$$

$$\therefore \alpha_1 - j\omega\alpha_2 = V_L(S)(S^2+\omega^2)\Big|_{s=-j\omega}$$

$$\alpha_1 - j\omega\alpha_2 = \frac{\left[\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega}{L_{DF}C}S\right](S^2+\omega^2)}{\left[(S+\frac{R}{2L})^2 + K_1^2\right]\left[S^2+\omega^2\right]}\Bigg|_{s=-j\omega}$$

$$\alpha_1 - j\omega\alpha_2 = \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega}{L_{DF}C}(-j\omega)}{\left[(-j\omega + \frac{R}{2L})^2 + K_1^2\right]}$$

$$= \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} + \frac{j\lambda_{DF}\omega^2}{L_{DF}C}}{j^2\omega^2 - j\frac{R\omega}{L} + \frac{R^2}{4L^2} + \frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\alpha_1 - j\omega\alpha_2 = \frac{L_{DF}E_m\omega - R\lambda_{DF}\omega + jL\lambda_{DF}\omega^2}{L_{DF}LC(-\omega^2 - j\frac{R\omega}{L} + \frac{1}{LC})}$$

$$= \frac{(L_{DF}E_m\omega - R\lambda_{DF}\omega) + jL\lambda_{DF}\omega^2}{L_{DF}(-LC\omega^2 - jRC\omega + 1)}$$

$$= \frac{(L_{DF}E_m\omega - R\lambda_{DF}\omega) + jL\lambda_{DF}\omega^2}{L_{DF}[(1-LC\omega^2) - jRC\omega]}$$

$$\alpha_1 - j\omega\alpha_2 = \frac{[(L_{DF}E_m\omega - R\lambda_{DF}\omega) + jL\lambda_{DF}\omega^2][(1-LC\omega^2) + jRC\omega]}{L_{DF}[(1-LC\omega^2) - jRC\omega][(1-LC\omega^2) + jRC\omega]}$$

$$\alpha_1 - j\omega\alpha_2 = \frac{(1-LC\omega^2)[(L_{DF}E_m\omega - R\lambda_{DF}\omega) + jL\lambda_{DF}\omega^2] + jRC\omega[(L_{DF}E_m\omega - R\lambda_{DF}\omega) + jL\lambda_{DF}\omega^2]}{L_{DF}[(1-LC\omega^2)^2 + (RC\omega)^2]}$$

$$= \frac{(1-LC\omega^2)(L_{DF}E_m\omega - R\lambda_{DF}\omega) + j(1-LC\omega^2)L\lambda_{DF}\omega^2 + jRC\omega(L_{DF}E_m\omega - R\lambda_{DF}\omega) - RC\omega^3L\lambda_{DF}}{L_{DF}[(1-LC\omega^2)^2 + (RC\omega)^2]}$$

$$= \frac{(1-LC\omega^2)(L_{DF}E_m\omega - R\lambda_{DF}\omega) - RC\omega^3L\lambda_{DF}}{L_{DF}[(1-LC\omega^2)^2 + (RC\omega)^2]} + j\frac{L\lambda_{DF}\omega^2(1-LC\omega^2) + RC\omega(L_{DF}E_m\omega - R\lambda_{DF}\omega)}{L_{DF}[(1-LC\omega^2)^2 + (RC\omega)^2]}$$

$$\alpha_1 = \frac{(1-LC\omega^2)(L_{DF}E_m\omega-R\lambda_{DF}\omega)-RC\omega^3L\lambda_{DF}}{L_{DF}[(1-LC\omega^2)^2+(RC\omega)^2]}$$

$$= \frac{L_{DF}E_m\omega-R\lambda_{DF}\omega-LC\omega^2(L_{DF}E_m\omega-R\lambda_{DF}\omega)-RC\omega^3L\lambda_{DF}}{L_{DF}[(1-LC\omega^2)^2+(RC\omega)^2]}$$

$$\alpha_1 = \frac{L_{DF}E_m\omega-R\lambda_{DF}\omega-LC\omega^3L_{DF}E_m+LC\omega^3R\lambda_{DF}-RC\omega^3L\lambda_{DF}}{L_{DF}[(1-LC\omega^2)^2+(RC\omega)^2]}$$

$$= \frac{L_{DF}E_m\omega-R\lambda_{DF}\omega-LC\omega^3L_{DF}E_m}{L_{DF}[(1-LC\omega^2)^2+(RC\omega)^2]}$$

$$\alpha_1 = \frac{L_{DF}E_m\omega(1-LC\omega^2)-R\lambda_{DF}\omega}{L_{DF}[(1-LC\omega^2)^2+(RC\omega)^2]}$$

or

$$\alpha_1 = \frac{E_m\omega(1-LC\omega^2)-R\omega(\frac{\lambda_{DF}}{L_{DF}})}{(1-LC\omega^2)^2+(RC\omega)^2}$$

$$-\omega\alpha_2 = \frac{L\lambda_{DF}\omega^2(1-LC\omega^2)+RC\omega(L_{DF}E_m\omega-R\lambda_{DF}\omega)}{L_{DF}[(1-LC\omega^2)^2+(RC\omega)^2]}$$

$$-\omega\alpha_2 = \frac{L\lambda_{DF}\omega^2(1-LC\omega^2)+RCL_{DF}E_m\omega^2-R^2C\omega^2\lambda_{DF}}{L_{DF}[(1-LC\omega^2)^2+(RC\omega)^2]}$$

$$-\alpha_2 = \frac{L\lambda_{DF}\omega - \lambda_{DF}\omega^3 L^2 C + RCL_{DF}E_m\omega - R^2 C\omega\lambda_{DF}}{L_{DF}[(1-LC\omega^2)^2 + (RC\omega)^2]}$$

$$= \frac{L\lambda_{DF}\omega(1-LC\omega^2) + R^2 C\omega\lambda_{DF} + RCL_{DF}E_m\omega}{L_{DF}[(1-LC\omega^2)^2 + (RC\omega)^2]}$$

$$-\alpha_2 = \frac{\lambda_{DF}[\omega L(1-LC\omega^2) - R^2 C\omega] + RCL_{DF}E_m\omega}{L_{DF}[(1-LC\omega^2) + (RC\omega)^2]}$$

$$\alpha_2 = \frac{\lambda_{DF}[R^2 C\omega - \omega L(1-LC\omega^2) - RCL_{DF}E_m\omega}{L_{DF}[(1-LC\omega^2) + (RCW)^2]}$$

or

$$\alpha_2 = \frac{\frac{\lambda_{DF}}{L_{DF}}[R^2 C\omega - \omega L(1-LC\omega^2)] - RCE_m\omega}{(1-LC\omega^2)^2 + (RC\omega)^2}$$

$$V_L(S)[(S+\frac{R}{2L})^2 + K_1^2]\Big|_{s=-\frac{R}{2L}-jK_1} = (\frac{\alpha_1+\alpha_2 S}{S^2+\omega^2})[(S+\frac{R}{2L})^2 + K_1^2]$$

$$+ \frac{\alpha_3+\alpha_4 S}{(S+\frac{R}{2L})^2 + K_1^2} \cdot [(S+\frac{R}{2L})^2 + K_1^2]$$

$$= \frac{\alpha_1+\alpha_2 S}{S^2+\omega^2}[(\frac{-R}{2L} - jK_1 + \frac{R}{2L})^2 + K_1^2] + \alpha_3 + \alpha_4 S$$

$$= \frac{\alpha_1+\alpha_2 S}{S^2+\omega^2}[j^2 K_1^2 + K_1^2] + \alpha_3 + \alpha_4(-\frac{R}{2L} - jK_1)$$

$$\alpha_3-\alpha_4 \frac{R}{2L} - j\alpha_4 K_1 = V_L(S)\left[(S+\frac{R}{2L})^2 + K_1^2\right]\Bigg|_{S = -\frac{R}{2L}-jK_1}$$

$$= \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega}{L_{DF}C}S]}{[(S+\frac{R}{2L})^2+K_1^2][S^2+\omega^2]} \cdot [(S+\frac{R}{2L})^2 + K_1^2]\Bigg|_{S = -\frac{R}{2L}-jK_1}$$

$$= \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega}{L_{DF}C}[-\frac{R}{2L}-jK_1]}{[(-\frac{R}{2L}-jK_1)^2+\omega^2]}$$

$$= \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} + \frac{\lambda_{DF}\omega R}{2LL_{DF}C} + j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}}{\frac{R^2}{4L^2} + j\frac{RK_1}{L} - K_1^2 + \omega^2}$$

$$\alpha_3-\alpha_4\frac{R}{2L} - j\alpha_4 K_1 = \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}}{L_{DF}LC} + \frac{\lambda_{DF}R\omega}{2LL_{DF}C} + j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}}{\frac{R^2}{4L^2} + j\frac{RK_1}{L} \; K_1^2+\omega^2}$$

$$= \frac{\frac{L_{DF}E_m\omega - R\lambda_{DF}\omega}{L_{DF}LC} + \frac{\lambda_{DF}\omega R}{2LL_{DF}C} + j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}}{\frac{R^2}{4L^2} + j\frac{RK_1}{L} - [\frac{1}{LC} - \frac{R^2}{4L^2}] + \omega^2}$$

$$= \frac{\frac{L_{DF}E_m\omega}{L_{DF}LC} - \frac{2R\lambda_{DF}\omega}{2L_{DF}LC} + \frac{\lambda_{DF}\omega R}{2LL_{DF}C} + j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}}{\frac{R^2}{4L^2} + j\frac{RK_1}{L} - \frac{1}{LC} + \frac{R^2}{4L^2} + \omega^2}$$

$$= \frac{[(\frac{L_{DF}E_m\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega R}{2LL_{DF}C}) + j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}]}{[(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}) + j\frac{RK_1}{L}]}$$

$$(\alpha_3-\alpha_4-\frac{R}{2L})-j\alpha_4K_1 = \frac{[(\frac{L_{DF}E_m\omega}{L_{DF}LC}-\frac{\lambda_{DF}\omega R}{2LL_{DF}C})+j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}][(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})-j\frac{RK_1}{L}]}{[(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})+j\frac{RK_1}{L}][(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})-j\frac{RK_1}{L}]}$$

$$\alpha_3-\alpha_4\frac{R}{2L}-j\alpha_4K_1 = \frac{(\frac{L_{DF}E_m\omega}{L_{DF}LC}-\frac{\lambda_{DF}\omega R}{2LL_{DF}C})[(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})-j\frac{RK_1}{L}]}{(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})^2+\frac{R^2K_1^2}{L^2}}$$

$$+\frac{j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}[(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})-j\frac{RK_1}{L}]}{(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})^2+\frac{R^2}{L^2}K_1^2}$$

$$\alpha_3-\alpha_4\frac{R}{2L}-j\,{}_4K_1 = \frac{(\frac{L_{DF}E_m\omega}{L_{DF}LC}-\frac{\lambda_{DF}\omega R}{2LL_{DF}C})(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC}) -j\frac{RK_1}{L}(\frac{L_{DF}E_m\omega}{L_{DF}LC}\ \frac{\lambda_{DF}\omega R}{2LL_{DF}C})}{(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})^2+\frac{R^2K_1^2}{L^2}}$$

$$+\frac{j\frac{\lambda_{DF}\omega K_1}{L_{DF}C}(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})+\frac{\lambda_{DF}\omega K_1^2R}{LL_{DF}C}}{(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC})^2+\frac{R^2K_1^2}{L_2}}$$

$$-\alpha_4 K_1 = \frac{-\frac{RK_1}{L}\left(\frac{L_{DF}E_m\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega R}{2LL_{DF}C}\right) + \frac{\lambda_{DF}\omega K_1}{L_{DF}C}\left(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}\right)}{\left(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}\right)^2 + \frac{R^2}{L^2}K_1^2}$$

$$\alpha_4 = \frac{\frac{R}{L}\left(\frac{L_{DF}E_m\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega R}{2LL_{DF}C}\right) - \frac{\lambda_{DF}\omega}{L_{DF}C}\left(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}\right)}{\left(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}\right)^2 + \frac{R^2}{L^2}K_1^2}$$

$$= \frac{\frac{RL_{DF}E_m\omega}{L^2L_{DF}C} - \frac{\lambda_{DF}\omega R^2}{2L^2L_{DF}C} - \frac{\lambda_{DF}\omega R^2}{2L^2L_{DF}C} - \frac{\lambda_{DF}\omega 3}{L_{DF}C} + \frac{\lambda_{DF}\omega}{L_{DF}C^2L}}{\left(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}\right)^2 + \frac{R^2}{L^2}K_1^2}$$

$$\alpha_4 = \frac{\frac{RL_{DF}E_m\omega}{L^2L_{DF}C} - \frac{\lambda_{DF}\omega R^2}{2L^2L_{DF}C} - \frac{\lambda_{DF}\omega R^2}{2L^2L_{DF}C} - \frac{\lambda_{DF}\omega^3}{L_{DF}C} + \frac{\lambda_{DF}\omega}{L_{DF}C^2L}}{\left(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}\right)^2 + \frac{R^2}{L^2}K_1^2}$$

$$\alpha_4 = \frac{\frac{RL_{DF}E_m\omega}{L^2L_{DF}C} - \frac{\lambda_{DF}\omega R^2}{L^2L_{DF}C} - \frac{\lambda_{DF}\omega^3}{L_{DF}C} + \frac{\lambda_{DF}\omega}{L_{DF}C^2L}}{\left(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}\right)^2 + \frac{R^2}{L^2}K_1^2}$$

$$= \frac{\frac{RE_m\omega}{L^2C} - \frac{\lambda_{DF}\omega R^2}{L^2L_{DF}C} - \frac{\lambda_{DF}\omega^3}{L_{DF}C} + \frac{\lambda_{DF}\omega}{L_{DF}C^2L}}{\frac{R^4}{4L^4} + \omega^4 + \frac{1}{L^2C^2} + \frac{2R^2\omega^2}{2L^2} - \frac{2R^2}{2L^3C} - \frac{2\omega^2}{LC} + \frac{R^2}{L^2}\left[\frac{1}{LC} - \frac{R^2}{4L^2}\right]}$$

$$= \frac{\frac{\omega}{L^2C}[RE_m - \frac{\lambda_{DF}R^2}{L_{DF}}] + \frac{\lambda_{DF}\omega}{L_{DF}C}[\frac{1}{LC} - \omega^2]}{\frac{R^4}{4L^2} + \omega^4 + \frac{1}{L^2C^2} + \frac{2R^2\omega^2}{2L^2} - \frac{2R^2}{2L^3C} - \frac{2\omega^2}{LC} + \frac{R^2}{L^3C} - \frac{R^4}{4L^4}}$$

$$\alpha_4 = \frac{\frac{\omega}{L^2C}[RE_m - \frac{\lambda_{DF}R^2}{L_{DF}}] + \frac{\lambda_{DF}\omega}{L_{DF}C}[\frac{1}{LC} - \omega^2]}{\omega^4 + \frac{1}{L^2C^2} - \frac{2\omega^2}{LC} + \frac{R^2\omega^2}{L^2}}$$

$$\alpha_4 = \frac{\frac{\omega}{L^2C}[RE_m - \frac{\lambda_{DF}R^2}{L_{DF}}] + \frac{\lambda_{DF}\omega}{L_{DF}C}[\frac{1}{LC} - \omega^2]}{(\omega^4 - \frac{2\omega^2}{LC} + \frac{1}{L^2C^2}) + \frac{2R^2\omega^2}{L^2}}$$

$$\alpha_4 = \frac{\frac{RE_m\omega}{L^2C} - \frac{\lambda_{DF}\omega R^2}{L^2L_{DF}C} + \frac{\lambda_{DF}\omega}{L_{DF}C}(\frac{1}{LC} - \omega^2)}{(\frac{1}{LC} - \omega^2)^2 + \frac{R^2\omega^2}{L^2}}$$

$$\alpha_4 = \frac{\frac{RE_m\omega}{L^2C} - \frac{\lambda_{DF}}{L_{DF}}[\frac{\omega R^2}{L^2C} - \frac{\omega}{C}(\frac{1}{LC} - \omega^2)]}{[\frac{1}{LC}(1-LC\omega^2)]^2 + \frac{R^2\omega^2}{L^2}}$$

$$\alpha_4 = \frac{\frac{RE_m\omega}{L^2C} - \frac{\lambda_{DF}}{L_{DF}}(\frac{\omega R^2}{L^2C}) + \frac{\omega\lambda_{DF}}{CL_{DF}}(\frac{1}{LC} - \omega^2)}{\frac{1}{L^2C^2}(1-LC\omega^2) + \frac{R^2\omega^2}{L^2}}$$

$$\alpha_4 = \frac{CRE_m\omega - \frac{\lambda_{DF}}{L_{DF}}(\omega R^2C) + \frac{\lambda_{DF}}{L_{DF}}(\omega L)(1-LC\omega^2)}{(1-LC\omega^2)^2 + R^2\omega^2C^2}$$

$$\alpha_4 = \frac{CRE_m\omega - \frac{\lambda_{DF}}{L_{DF}}[R^2C\omega-\omega L(1-LC\omega^2)]}{(1-LC\omega^2)^2 + (R\omega C)^2}$$

and

$$\alpha_3-\alpha_4(\frac{R}{2L}) = \frac{(\frac{L_{DF}E_m\omega}{L_{DF}LC} - \frac{\lambda_{DF}\omega R}{2LL_{DF}C})(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC}) + \frac{\lambda_{DF}\omega K_1^2 R}{LL_{DF}C}}{(\frac{R^2}{2L^2} + \omega^2 - \frac{1}{LC})^2 + \frac{R^2}{L^2}K_1^2}$$

$$= \frac{\frac{L_{DF}E_m\omega}{L_{DF}LC}(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC}) - \frac{\lambda_{DF}\omega R}{2LL_{DF}C}(\frac{R^2}{2L^2}+\omega^2-\frac{1}{LC}) + \frac{\lambda_{DF}\omega K_1^2 R}{LL_{DF}C}}{\frac{1}{L^2C^2}(1-LC\omega^2)^2 + \frac{R^2\omega^2}{L^2}}$$

$$= \frac{L_{DF}E_m\omega R^2}{2L^3L_{DF}C} + \frac{L_{DF}E_m\omega^3}{L_{DF}LC} - \frac{L_{DF}E_m\omega}{L_{DF}L^2C^2} - \frac{\lambda_{DF}\omega R^3}{4L^3L_{DF}C}$$

$$\alpha_3-\alpha_4\,\frac{R}{2L} = \frac{- \frac{\lambda_{DF}R\omega^3}{2LL_{DF}C} + \frac{\lambda_{DF}\omega R}{2L^2C^2L_{DF}} + \frac{\lambda_{DF}\omega K_1^2 R}{LL_{DF}C}}{\frac{1}{L^2C^2}(1-LC\omega^2) + \frac{R^2\omega^2}{L^2}}$$

$$\alpha_3-\alpha_4\,\frac{R}{2L} = \frac{\frac{L_{DF}E_m\omega R^2}{2L^3L_{DF}C} + \frac{L_{DF}E_m\omega^3}{L_{DF}LC} - \frac{L_{DF}E_m\omega}{L_{DF}L^2C^2} - \frac{\lambda_{DF}\omega R^3}{4L^3L_{DF}C} - \frac{\lambda_{DF}R\omega^3}{2LL_{DF}C} + \frac{\lambda_{DF}\omega R}{2L^2C^2L_{DF}} + \frac{\lambda_{DF}\omega R}{L_{DF}LC}[\frac{1}{LC} - \frac{R^2}{4L^2}]}{\frac{1}{L^2C^2}(1-LC\omega^2)^2 + \frac{R^2\omega^2}{L^2}}$$

$$\alpha_3-\alpha_4\frac{R}{2L} = \frac{\frac{L_{DF}E_m\omega R^2}{2L^3L_{DF}C}+\frac{L_{DF}E_m\omega^3}{L_{DF}LC}-\frac{L_{DF}E_m\omega}{L_{DF}L^2C^2}-\frac{\lambda_{DF}\omega R^3}{2L^3L_{DF}C}-\frac{\lambda_{DF}R\omega^3}{2LL_{DF}C}+\frac{3\lambda_{DF}\omega R}{2L^2C^2L_{DF}}}{\frac{1}{L^2C^2}(1-LC\omega^2)^2+\frac{R^2\omega^2}{L^2}}$$

$$\therefore \alpha_3 = \frac{E_m[\frac{CR^2\omega}{L}+(1-LC\omega^2)]-\frac{\lambda_{DF}}{L_{DF}}(\frac{R^3C\omega}{L}-2\omega R+\omega^3RCL)}{(1-LC\ ^2)^2+R^2C^2\omega^2}$$

$$V_L(S) = \frac{\alpha_1+\alpha_2S}{S^2+\omega^2}+\frac{\alpha_3+\alpha_4S}{(S+\frac{R}{2L})^2+K_1^2}$$

$$V_L(S) = \frac{\alpha_1}{S^2+\omega^2}+\frac{\alpha_2S}{S^2+\omega^2}+\frac{\alpha_3}{(S+\frac{R}{2L})^2+K_1^2}+\frac{\alpha_4S}{(S+\frac{R}{2L})^2+K_1^2}$$

$$= \frac{\alpha_1}{\omega}(\frac{\omega}{S^2+\omega^2})+\alpha_2(\frac{S}{S^2+\omega^2})+\frac{\alpha_3}{K_1}[\frac{K_1}{(S+\frac{R}{2L})^2+K_1^2}]$$

$$+\alpha_4[\frac{S+\frac{R}{2L}-\frac{R}{2L}}{(S+\frac{R}{2L})^2+K_1^2}]$$

$$= \frac{\alpha_1}{\omega}(\frac{\omega}{S^2+\omega^2})+\alpha_2(\frac{S}{S^2+\omega^2})+\frac{\alpha_3}{K_1}[\frac{K_1}{(S+\frac{R}{2L})^2+K_1^2}]+\alpha_4[\frac{S+\frac{R}{2L}}{(S+\frac{R}{2L})+K_1^2}]$$

$$-\frac{\alpha_4R}{2LK_1}\frac{K_1}{(S+\frac{R}{2L})^2+K_1^2}$$

$$\mathcal{L}^{-1}V_L(S) = V_L(t)$$

$$V_L(t) = \frac{\alpha_1}{\omega}\sin \omega t + \alpha_2\cos \omega t + \frac{\alpha_3}{K_1}e^{-\frac{R}{2L}t}\sin K_1t$$

$$+ \alpha_4 e^{-\frac{R}{2L}t}\cos K_1t - \frac{\alpha_4 R}{2LK_1} e^{-\frac{R}{2L}t}\sin K_1t$$

$$V_L(t) = \frac{\alpha_1}{\omega}\sin \omega t + \alpha_2\cos \omega t + \left(\frac{\alpha_3}{K_1} - \frac{\alpha_4 R}{2LK_1}\right)e^{-\frac{R}{2L}t}\sin K_1t$$

$$+ \alpha_4 e^{-\frac{R}{2L}t}\cos K_1t$$

or

$$V_L(t) = \frac{\alpha_1}{\omega}\sin \omega t + \alpha_2\cos \omega t + \gamma e^{-\frac{R}{2L}t}\sin K_1t + \alpha_4 e^{-\frac{R}{2L}t}\cos K_1t$$

$$V_L(t) = \sqrt{\left(\frac{\alpha_1}{\omega}\right)^2+\alpha_2^2}\left[\frac{\frac{\alpha_1}{\omega}}{\sqrt{\left(\frac{\alpha_1}{\omega}\right)^2+\alpha_2^2}}\sin \omega t + \frac{\alpha_2}{\sqrt{\left(\frac{\alpha_1}{\omega}\right)^2+\alpha_2^2}}\cos \omega t\right]$$

$$+ \sqrt{\gamma^2+\alpha_4^2}\left[\frac{\gamma}{\sqrt{\gamma^2+\alpha_4^2}}\sin K_1t + \frac{\alpha_4}{\sqrt{\gamma^2+\alpha_4^2}}\cos K_1t\right]e^{-\frac{R}{2L}t}$$

$$V_L(t) = \sqrt{\left(\frac{\alpha_1}{\omega}\right)^2 + \alpha_2^2}\,[\cos\phi_1 \sin\omega t + \sin\phi_1 \cos\omega t]$$

$$+ \sqrt{\gamma^2+\alpha_4^2}\,[\cos\phi_2 \sin K_1 t + \sin\phi_2 \cos K_1 t]$$

$$V_L(t) = \sqrt{\left(\frac{\alpha_1}{\omega}\right)^2 + \alpha_2^2}\,[\sin(\omega t + \phi_1)] + \sqrt{\gamma^2+\alpha_4^2}\; e^{-\frac{R}{2L}t}\,[\sin(K_1 t+\phi_2)]$$

Steady state solution:

$$V_L(t) = \sqrt{\left(\frac{\alpha_1}{\omega}\right)^2 + \alpha_2^2}\,\sin(\omega t + \phi_1) = A\sin(\omega t + \phi_1)$$

where

$$\phi_1 = \tan^{-1}\frac{\alpha_2}{\frac{\alpha_1}{\omega}}$$

$$\phi_1 = \tan^{-1}\frac{\alpha_2 \omega}{\alpha_1}$$

and

$$A = \sqrt{\left(\frac{\alpha_1}{\omega}\right)^2 + \alpha_2^2}$$

$$A^2 = \left(\frac{\alpha_1}{\omega}\right)^2 + \alpha_2^2$$

$$\alpha_1 = \frac{E_m\omega(1-LC\omega^2) - R\omega\left(\frac{\lambda_{DF}}{L_{DF}}\right)}{(1-LC\omega^2)^2+R^2C^2\omega^2}$$

$$\alpha_2 = \frac{\frac{\lambda_{DF}}{L_{DF}}[R^2C\omega-\omega L(1-LC\omega^2)]-RCE_m\omega}{(1-LC\omega^2)^2 + R^2C^2\omega^2}$$

$$\frac{\alpha_1}{\omega} = \frac{E_m(1-LC\omega^2)-R(\frac{\lambda_{DF}}{L_{DF}})}{(1-LC\omega^2)^2+R^2C^2\omega^2}$$

$$A^2 = \frac{\alpha_1^2}{\omega^2} + \alpha_2^2$$

$$A = \frac{\sqrt{[E_m(1-LC\omega^2)-R(\frac{\lambda_{DF}}{L_{DF}})]^2+[(\frac{\lambda_{DF}}{L_{DF}})\ R^2C\omega-\omega L(1-LC\omega^2)\ -RCE_m\omega]}}{(1-LC\omega^2)^2+R^2C^2\omega^2}$$

$$\phi_1 = \tan^{-1} \frac{\omega\frac{\lambda_{DF}}{L_{DF}}[R^2C\omega-\omega L(1-LC\omega^2)]-RCE_m\omega^2}{E_m\omega(1-LC\omega^2)-R\omega(\frac{\lambda_{DF}}{L_{DF}})}$$

$$\phi_1 = \tan^{-1} \frac{\frac{\lambda_{DF}}{L_{DF}}[R^2C\omega-\omega L(1-LC\omega^2)]-RCE_m\omega^2}{E_m(1-LC\omega^2)-R(\frac{\lambda_{DF}}{L_{DF}})}$$

$$\phi_2 = \tan^{-1}\frac{\alpha_4}{\gamma} = \tan^{-1}\frac{2LK_1\alpha_4}{2L\alpha_3-\alpha_4 R}$$

Total solution:

$$V_L(t) = A\sin(\omega t+\phi_1) + Be^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)$$

where

$$B = \sqrt{\gamma^2+\alpha_4^2}$$

and

$$\gamma = \frac{\alpha_3}{K_1} - \frac{\alpha_4 R}{2LK_1} = \frac{2L\alpha_3 - \alpha_4 R}{2LK_1}$$

$$\alpha_3 = \frac{E_m[\frac{CR^2\omega}{L} + \omega(1-LC\omega^2)] - \frac{\lambda_{DF}}{L_{DF}}(\frac{R^3C\omega}{L} - 2\omega R + \omega^3 RCL)}{(1-LC\omega^2) + R^2C^2\omega^2}$$

$$\alpha_4 = \frac{CRE_m\omega - \frac{\lambda_{DF}}{L_{DF}}[R^2C\omega - \omega L(1-LC\omega^2)]}{(1-LC\omega^2)^2 + (R\omega C)^2}$$

$$B = \sqrt{(\frac{2L\alpha_3 - \alpha_4 R}{2LK_1})^2 + \alpha_4^2}$$

Since

$$\frac{1}{c}\int i_t dt = \frac{d\lambda(t)}{dt} = V_L(t)$$

$$i_c = C\frac{dV_L(t)}{dt}$$

$$i_c = C\frac{d}{dt}[A\sin(\omega t+\phi_1) + Be^{-\frac{R}{2L}t}\sin(K_1 t+\phi_2)]$$

$$i_c = c[A\omega\cos(\omega t+\phi_1) + B\{-(\frac{R}{2L})e^{-\frac{R}{2L}t}\sin(K_1 t+\phi_2)$$
$$+ e^{-\frac{R}{2L}t}K_1\cos(K_1 t+\phi_2)\}]$$

$$i_c = C\omega A\cos(\omega t+\phi_1) - CB\frac{R}{2L}e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)$$

$$+ BK_1Ce^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)$$

since

$$i_L = \frac{1}{L_{DF}}\int V_L(t)dt + i_L(0)$$

$$i_L = \frac{1}{L_{DF}}\int_0^t [\underbrace{A\sin(\omega t+\phi_1)}_{I} + \underbrace{Be^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)}_{II}]dt + i_L(0)$$

$$I = \frac{1}{L_{DF}}\int_0^t A\ \sin(\omega t+\phi_1)$$

$$I = \frac{1}{L_{DF}}[\frac{-A\cos(\omega t+\phi_1)}{\omega}]_0^t = - \frac{A}{L_{DF}\omega}[\cos(\omega t+\phi_1) - \cos\phi_1]$$

$$I = - \frac{A}{L_{DF}\omega}[\cos(\omega t+\phi_1) - \cos\ \phi_1]$$

$$II = \frac{B}{L_{DF}}\int_0^t e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)dt \quad \text{Integrating by parts}$$

$$u = \sin(K_1t+\phi_2) \qquad \text{and} \qquad dv = e^{-\frac{R}{2L}t}dt$$

$$\frac{du}{dt} = K_1\cos(K_1t+\phi_2) \qquad V = \frac{e^{-\frac{R}{2L}t}}{-\frac{R}{2L}} = -\frac{2L}{R}e^{-\frac{R}{2L}t}$$

$$= \frac{B}{L_{DF}}[-\frac{2L}{R}e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)$$

$$- \int_0^t -\frac{2L}{R}e^{-\frac{R}{2L}t}K_1\cos(K_1t+\phi_2)dt]$$

$$II = -\frac{2BL}{RL_{DF}}e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2) +$$

$$+ \frac{2LBK_1}{L_{DF}R}\int_0^t e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)dt$$

$$dV = e^{-\frac{R}{2L}t}dt \qquad u = \cos(K_1t+\phi_2)$$

$$V = \frac{e^{-\frac{R}{2L}t}}{-\frac{R}{2L}} = -\frac{2L}{R}e^{-\frac{R}{2L}t} \qquad du = -K_1\sin(K_1t+\phi_2)dt$$

$$III = \frac{2LBK_1}{L_{DF}R}[-\frac{2L}{R}e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2) - \int_0^t -\frac{2L}{R}e^{-\frac{R}{2L}t}K_1 \times$$

$$\times (-\sin(K_1t+\phi_2))dt]$$

$$III=\frac{-4L^2BK_1}{L_{DF}R^2}\, e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2) - \frac{4L^2BK_1^2}{R^2L_{DF}}\int_0^t e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)\,dt$$

$$II = -\frac{2BL}{RL_{DF}}e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)-\frac{4BL^2K_1}{L_{DF}R^2}\, e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)$$
$$-\frac{4L^2BK_1}{L_{DF}R^2}\int_0^t e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)\,dt$$

But

$$II = \frac{B}{L_{DF}}\int_0^t e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)\,dt$$

$$\left(\frac{B}{L_{DF}} + \frac{4L^2BK_1^2}{L_{DF}R^2}\right)\int_0^t e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)\,dt$$
$$=\frac{-2BL}{RL_{DF}}e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)$$
$$-\frac{4BL^2K_1}{L_{DF}R^2}\, e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)$$

or

$$\int_0^t e^{-\frac{R}{2L}t}\sin(K_1t+\phi_1)\,dt = \frac{-\frac{2BL}{R}\ L_{DF}}{\left(\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}\right.} e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)$$

$$-\frac{\frac{4BL^2K_1}{L_{DF}R^2}}{\left(\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}\right)}\, e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)$$

$$i_L(t) = I + II$$

$$i_L(t) = \frac{-A}{L_{DF}\omega}[\cos(\omega t+\phi_1)-\cos\ \phi_1]$$

$$+ \frac{B}{L_{DF}}\left[\frac{\frac{-2BL}{RL_{DF}}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right] e^{-\frac{R}{2L}t} \sin(K_1 t+\phi_2)$$

$$- \frac{B}{L_{DF}}\left[\frac{\frac{4BL^2K_1}{L_{DF}R^2}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right] e^{-\frac{R}{2L}t} \cos(K_1 t+\phi_2)$$

$$- \left(\frac{-\frac{2B^2L}{RL^2{}_{DF}}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right)\sin\ \phi_2 + \left(\frac{\frac{4B^2L^2K_1}{L^2{}_{DF}R^2}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right)\cos\ \phi_2$$

$$i_L(t) = -\frac{A}{L_{DF}\omega}[\cos(\omega t+\phi_1)-\cos\ \phi_1]$$

$$+ \frac{B}{L_{DF}}\left[\frac{-2BRL}{R^2B+4L^2BK_1}\right] e^{-\frac{R}{2L}t} \sin(K_1 t+\phi_2)$$

$$+ \frac{B}{L_{DF}}\left[\frac{4BL^2K_1}{BR^2+4L^2BK_1}\right] e^{-\frac{R}{2L}t} \cos(K_1 t+\phi_2)$$

$$- \left(\frac{-\frac{2B^2L}{RL^2{}_{DF}}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right)\sin\ \phi_2 + \left(\frac{\frac{4B^2L^2K_1}{L^2{}_{DF}R^2}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right)\cos\ \phi_2$$

$$i_L(t) = -\frac{A}{L_{DF}\omega}[\cos(\omega t+\phi_1)-\cos\phi_1] - \frac{1}{L_{DF}}[\frac{2BRL}{R^2+4L^2K_1}]e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2) \quad (13.5)$$

$$-\frac{1}{L_{DF}}[\frac{4L^2K_1B}{R^2+4L^2K_1}]e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)$$

$$-\left(\frac{-\frac{2B^2L}{RL^2_{DF}}}{\frac{B}{L_{DF}}+\frac{4L^2BK_1}{L_{DF}R^2}}\right)\sin\phi_2 + \left(\frac{\frac{4B^2L^2K_1}{L^2_{DF}R^2}}{\frac{B}{L_{DF}}+\frac{4L^2BK_1}{L_{DF}R^2}}\right)\cos\phi_2$$

$$i = i_c + i_L$$

$$i = C\omega A\cos(\omega t+\phi_1) - \frac{CBR}{2L}e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)$$

$$+ BK_1Ce^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)$$

$$-\frac{A}{L_{DF}\omega}[\cos(\omega t+\phi_1)-\cos\phi_1]$$

$$-\frac{1}{L_{DF}}[\frac{2BR}{R^2+4L^2K_1}]e^{-\frac{R}{2L}t}\sin(K_1t+\phi_2)$$

$$-\frac{1}{L_{DF}}[\frac{4L^2K_1B}{R^2+4L^2K_1}]e^{-\frac{R}{2L}t}\cos(K_1t+\phi_2)$$

$$-\left(\frac{-\frac{2B^2L}{RL^2_{DF}}}{\frac{B}{L_{DF}}+\frac{4L^2BK_1}{L_{DF}R^2}}\right)\sin\phi_2 + \left(\frac{\frac{4B^2L^2K_1}{L^2_{DF}R^2}}{\frac{B}{L_{DF}}+\frac{4L^2BK_1}{L_{DF}R^2}}\right)\cos\phi_2$$

$$i = C\omega A\cos(\omega t+\phi_1) - \frac{A}{L_{DF}\omega}[\cos(\omega t+\phi_1) - \cos\phi_1]$$

$$- e^{-\frac{R}{2L}t}\sin(K_1 t+\phi_2)\left[\frac{CBR}{2L} + \frac{2BRL}{L_{DF}(R^2+4L^2K_1}\right]$$

$$+ e^{-\frac{R}{2L}t}\cos(K_1 t+\phi_2)\left[\frac{4LK_1B}{L_{DF}(R^2+4L^2K_1} + BK_1C\right.$$

$$- \left(\frac{-\frac{2B^2L}{RL^2_{DF}}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right)\sin\phi_2 + \left(\frac{\frac{4B^2L^2K_1}{L^2_{DF}R^2}}{\frac{B}{L_{DF}} + \frac{4L^2BK_1}{L_{DF}R^2}}\right)\cos\phi_2 \qquad (13.6)$$